

# DISTRIBUTION OF RATIONAL MAPS WITH A PREPERIODIC CRITICAL POINT

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*Dedicated to the memory of Adrien Douady*

**ABSTRACT.** Let  $\{f_\lambda\}$  be any algebraic family of rational maps of a fixed degree, with a marked critical point  $c(\lambda)$ . We first prove that the hypersurfaces of parameters for which  $c(\lambda)$  is periodic converge as a sequence of positive closed  $(1, 1)$  currents to the bifurcation current attached to  $c$  and defined by DeMarco [DeM1]. We then turn our attention to the parameter space of polynomials of a fixed degree  $d$ . By intersecting the  $d - 1$  currents attached to each critical point of a polynomial, Bassanelli and Berteloot [BB1] obtained a positive measure  $\mu_{\text{bif}}$  of finite mass which is supported on the connectedness locus. They showed that its support is included in the closure of the set of parameters admitting  $d - 1$  neutral cycles. We show that the support of this measure is precisely the closure of the set of strictly critically finite polynomials (i.e. of Misiurewicz points).

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## INTRODUCTION

It is a central problem in dynamics to understand how the dynamics of a map can change under perturbation. In the context of rational maps of the Riemann sphere, this question has received a lot of attention, and is still a very active area of research. The seminal paper of Mañé-Sad-Sullivan [MSS] has first paved the way to understand structural stability of general holomorphic dynamical systems, and was completed a few years later by the construction of a Teichmüller theory for rational maps in [McMS]. Besides these general results, many particular families have been studied in detail. Among them are the family of quadratic polynomials [DH, Man], the set of cubic polynomials [BrH1, BrH2, K3], or the space of quadratic rational maps [Re1, Re2, Mi1].

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Let  $(\Lambda, f) = \{f_\lambda, \lambda \in \Lambda\}$  be a holomorphic family of rational maps of degree  $d \geq 2$ , parameterized by a smooth connected complex manifold  $\Lambda$  (of any dimension). We say that a critical point is *marked* if it can be followed holomorphically along  $\Lambda$  (see Section 2.1 for precise definitions). Following the terminology of McMullen [McM1], a critical point is *passive* at  $\lambda_0 \in \Lambda$  if the family  $\{f_\lambda^n c(\lambda)\}_n$  is normal in some neighborhood of  $\lambda_0$ . Otherwise,  $c$  is *active*. It follows from [MSS] that a rational map is unstable if and only if at least one of its critical points is active.

Following L.DeMarco [DeM1] it is possible to associate a natural positive closed (1,1) current  $T$  to a marked critical point  $\lambda \mapsto c(\lambda)$ . The support of  $T$  is the *activity locus* of  $c(\lambda)$ . Her method requires lifting the situation to  $\mathbb{C}^2 \setminus \{0\}$ . Our first aim is to present a coordinate free presentation of her results –see Proposition-Definition 3.1 and Theorem 3.2 below. Our presentation is quite classical in higher dimensional holomorphic dynamics and owes much to the work of V.Guedj [G1, G2], see also [DS].

We then study the distribution of parameters for which  $c(\lambda)$  is (pre-)periodic. A classical normal families argument using Montel's Theorem implies that if  $c(\lambda)$  is active at  $\lambda_0$ , then arbitrarily close to  $\lambda_0$  there are parameters for which  $c(\lambda)$  is preperiodic. We prove a *quantitative* version of this fact under the form of an equidistribution theorem, which is a parameter space analog of the following classical theorem: the periodic points of a fixed rational map equidistribute towards the maximal entropy measure, see [To, Ly1].

For any  $n > k \geq 0$ , let us introduce the set  $\text{Per}(n, k)$  of parameters for which  $f_\lambda^n c(\lambda) = f_\lambda^k c(\lambda)$ . It is either equal to the whole variety  $\Lambda$  or it is a hypersurface. In the former case, we call the family *trivial*: the activity locus is then empty, and the current  $T$  equals 0. In the latter case, we consider  $[\text{Per}(n, k)]$  the current of integration over the divisor  $\text{Per}(n, k)$  (each component is counted with its multiplicity as a solution of  $f_\lambda^n c(\lambda) = f_\lambda^k c(\lambda)$ ).

**Theorem 1.** *Let  $(\Lambda, f, c)$  be a non-trivial holomorphic family of rational maps of  $\mathbb{P}^1$  of degree  $d \geq 2$  with a marked critical point. If  $\Lambda$  is quasi-projective, then for any sequence of integers  $0 \leq k(n) < n$ , the following convergence statement holds:*

$$\lim_{n \rightarrow \infty} \frac{[\text{Per}(n, k(n))]}{d^n + d^{(1-e)k(n)}} = T$$

where  $e \in \{0, 1\}$  is the cardinality of the exceptional set of a generic map  $f_\lambda$ .

In other words,  $e = 0$  if, for some (hence for a generic)  $\lambda$ ,  $f_\lambda$  is not Möbius-conjugate to a polynomial. On the other hand,  $e = 1$  means that all  $f_\lambda$  are polynomials. We refer to Section 2.3 for a brief discussion on exceptional points in families of rational maps.

We also prove (Theorem 2.5) that, except for some trivial cases, in a family of rational maps parameterized by a quasiprojective manifold, the current  $T$  is always non trivial. See the comments following Theorem 4 for more details.

A natural method for proving the theorem is the following. We already mentioned that for any rational map  $f$ , periodic points equidistribute towards the measure of maximal entropy. These results can be put together to yield a convergence theorem in  $\Lambda \times \mathbb{P}^1$ . There exists a positive closed  $(1, 1)$  current  $\widehat{T}$  on  $\Lambda \times \mathbb{P}^1$  such that  $d^{-n}[\{(\lambda, z), f_\lambda^n(z) = z\}] \rightarrow \widehat{T}$ . We can now look at the hypersurface  $\Gamma := \{(\lambda, c(\lambda))\} \subset \Lambda \times \mathbb{P}^1$ . The projection  $\pi$  onto the first factor gives an analytic isomorphism from  $\Gamma$  to  $\Lambda$ , and the constructions are done in such a way that  $T = \pi_*(\widehat{T}|_\Gamma)$ . Observe that  $d^{-n}[\{(\lambda, z), f_\lambda^n(z) = z\}]|_\Gamma \rightarrow \widehat{T}|_\Gamma$  is equivalent to the statement of our theorem. However, it is important to note that the convergence of currents is not strong enough to ensure the convergence on  $\Gamma$ . Our result thus needs a separate argument.

Our proof relies on potential theory, and bears some similarity with the argument designed by Brolin for the proof of his celebrated equidistribution theorem [B]. This results in the appearance of the assumption of quasiprojectivity, because we use the maximum principle for plurisubharmonic functions in the course of the proof.

The theorem applies immediately in a number of interesting examples, including the family of polynomials of degree  $d$  (Section 6), and the family of rational maps of degree  $d$  (Section 8). In the latter family, Bassanelli-Berteloot have recently used the description of the bifurcation current in terms of Lyapunov exponents to prove equidistribution results of parameters admitting a periodic cycle with a fixed multiplier, see [BB2].

By applying our convergence theorem to the one-parameter family of unicritical polynomials of degree  $d$ , we get the following corollary which is due to Levin [Le2].

**Corollary 2** ([Le2]). *Let  $\mathcal{M}_d$  be the set of complex numbers  $c$  for which  $f_c(z) = z^d + c$  has connected Julia set (the Mandelbrot set). Then*

$$\lim_{n \rightarrow \infty} \frac{1}{d^n} \sum_{f_c^n(0)=0} \delta_c = \mu$$

where  $\mu$  is the harmonic measure on  $\mathcal{M}_d$ .

It is remarkable that this statement can be obtained by arithmetic methods based on height theory. It is a consequence of [A], and it was explicitly stated (when  $d = 2$ ) in [BaH, Theorem 8.13]. Using height theory presents the advantage to yield a precise estimate on the speed of convergence. For instance, a proof of the following result is given in [FRL].

**Theorem 3** ([FRL]). *With notation as in Corollary 2, let  $F_n \subset \mathbb{C}$  be a sequence of disjoint finite sets, invariant under the absolute Galois group of  $\mathbb{Q}$ , and included in the union  $\bigcup_{n \neq k} \{f_c^n(0) = f_c^k(0)\}$ . Then for any compactly supported  $C^1$  function  $\varphi$ , we have*

$$\left| \frac{1}{|F_n|} \sum_{c \in F_n} \varphi(c) - \int \varphi d\mu \right| \leq C \left( \frac{\log |F_n|}{|F_n|} \right)^2 \times \sup\{|\varphi|, |\varphi'|\},$$

where  $|F|$  denotes the cardinality of  $F$ .

Unfortunately, it is not clear how to extend this method to higher dimensional parameter spaces.

Another issue in the proof of the convergence Theorem 1 is whether preperiodic critical points are active or not. Of course if a critical point is preperiodic to a repelling cycle it is

active, and if it is preperiodic to an attracting cycle it is passive. In the neutral case, if for instance  $\Lambda$  is the space of all rational maps (or all polynomials), such a critical point can directly be perturbed to become prerepelling, so it is active. In the case of a general family  $\Lambda$ , the situation is more delicate, and it seems that this question, although natural, has not been previously addressed.

**Theorem 4.** *Let  $(\Lambda, f, c)$  be any holomorphic family of rational maps of  $\mathbb{P}^1$  of degree  $d \geq 2$  with a marked critical point. Assume  $U \subset \Lambda$  is a connected open subset where  $c$  is passive. Then exactly one of the following cases hold.*

- (1)  *$c$  is never preperiodic in  $U$ . In this case the closure of the orbit of  $c$  can be followed by a holomorphic motion.*
- (2)  *$c$  is persistently preperiodic in  $U$ .*
- (3) *The set of parameters for which  $c$  is preperiodic is a closed subvariety in  $U$ . Moreover, either there exists a persistently attracting (possibly superattracting) cycle attracting  $c$  throughout  $U$ , or  $c$  lies in the interior of the linearization domain associated to a persistent irrationally neutral periodic point.*

This result is in fact a consequence of a purely local statement, Theorem 1.1. As a consequence of these techniques, we give in Section 2.2 the following generalization of a theorem of McMullen [McM2]: for any *algebraic* family of rational maps with a marked critical point  $(\Lambda, f, c)$ , either  $f$  is constant, or  $c$  is persistently preperiodic, or the activity locus of  $c$  is non empty (see Theorem 2.5 for a precise statement).

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When all critical points are marked, the bifurcation locus is the union of the various activity loci of the critical points. This can be refined by introducing successive bifurcation loci, indexed by the number of critical points being active.

A natural expectation is that if  $k$  marked critical points are active at  $\lambda_0$ , there should be a nearby parameter where the  $k$  critical points are preperiodic. This is of particular interest when  $k$  is maximal, so that the perturbed map becomes critically finite. However, there is no reasonable analogue of Montel's Theorem for sequences of holomorphic mappings in higher dimension (due to the Fatou-Bieberbach phenomenon), and as it turns out, this expectation is wrong. Indeed there exist cubic polynomials with both critical points active such that for every perturbation, one of the two critical points is attracted by an attracting cycle (see Example 6.13). We thank A. Douady for kindly communicating this example to us.

It is one of the main ideas in higher dimensional holomorphic dynamics that pluripotential theory and the use of positive closed currents can serve as a natural substitute to Montel's Theorem. Later on we shall see that in order to obtain a correct statement, we need to replace the locus where  $k$  critical points are active by the support of the wedge product of  $k$  suitable bifurcation currents.

In section 6, we implement this strategy in the space  $\mathcal{P}_d$  of polynomials of degree  $d$  with all critical points marked, which, up to finite branched cover, is biholomorphic to  $\mathbb{C}^{d-1}$ . In this space we can consider  $d-1$  positive closed currents of bidegree  $(1,1)$ :  $T_0, \dots, T_{d-2}$ , associated to the marked critical points, as well as the so called *bifurcation current*  $T_{\text{bif}} = \sum T_i/(d-1)$  introduced by DeMarco [DeM1] and its successive powers  $(T_{\text{bif}})^{\wedge k}$ ,  $k = 1, \dots, d-1$  introduced by Bassanelli and Berteloot [BB1]. Although both papers are concerned with the more difficult situation of rational maps of degree  $d$ , their results apply equally in the context of polynomial maps (see also [Ph] for related definitions in a wider context).

The support of  $(T_{\text{bif}})^{\wedge k}$  is contained in the set of parameters where  $k$  critical points are active. This inclusion can be strict, as shown by the already mentioned Example 6.13. Basanelli and Berteloot proved that near any point in  $\text{Supp}(T_{\text{bif}}^{\wedge k})$ , there is a parameter with  $k$  neutral periodic orbits. Here, we obtain the following result:

**Theorem 5.** *There exists a sequence of codimension  $k$  algebraic subvarieties  $W_n$  (not necessarily reduced) and positive real numbers  $\alpha_n$  such that  $\alpha_n[W_n]$  converges to  $(T_{\text{bif}})^{\wedge k}$ , and  $W_n$  is supported in the set of parameters for which  $k$  critical points are preperiodic.*

In particular near every  $\lambda \in \text{Supp}(T_{\text{bif}}^{\wedge k})$  there is a parameter for which  $k$  critical points are preperiodic. Theorem 5 is proved by successive applications of Theorem 1 on the subvarieties where a fixed number of critical points is periodic with a fixed period.

The case where  $k = d - 1$  deserves special attention. In this case the suitably normalized positive measure  $\mu_{\text{bif}} := c(T_{\text{bif}})^{\wedge d-1}$  is a probability measure, supported on the boundary of the connectedness locus, which is compact in  $\mathcal{P}_d$ . Recall that a polynomial is said to be of *Misiurewicz type* if all its critical points are preperiodic to repelling cycles. It is classical that if all critical points are strictly preperiodic, then the Misiurewicz property holds. From Theorem 5 we immediately get the following corollary.

**Corollary 6.** *The support of the bifurcation measure  $\mu_{\text{bif}}$  is contained in the closure of Misiurewicz parameters.*

We obtain several interesting characterizations of the bifurcation measure, which are well known when  $d = 2$  (and more generally for unicritical polynomials). First, when  $d = 2$ ,  $\mu_{\text{bif}}$  is the harmonic measure of the Mandelbrot set. To give a precise statement in higher dimension, we need to use a parameterization of  $\mathcal{P}_d$  by the affine space  $\mathbb{C}^{d-1}$  (see Section 5 for more details).

**Proposition 7.** *The bifurcation measure is the pluricomplex equilibrium measure of the connectedness locus  $\mathcal{C} \subset \mathbb{C}^{d-1}$ . As a consequence,  $\text{Supp}(\mu_{\text{bif}})$  is the Shilov boundary of  $\mathcal{C}$ .*

Being the pluriharmonic measure of the connectedness locus shows that this measure is natural from the point of view of complex analysis.

Our next result is a characterization of the measure  $\mu_{\text{bif}}$  in  $\mathcal{P}_d$  as the landing measure of a family of external rays. Let us explain how these rays are defined. Let  $P$  be a polynomial whose Green function takes the same value  $r > 0$  at all critical points. The set  $\Theta$  of external angles of rays landing at the critical points gives a natural way to describe the combinatorics of  $P$ , see [Go, BFH, K2] where  $\Theta$  is referred to as the *critical portrait* of  $P$ . Now, we may deform  $P$  in the shift locus by leaving  $\Theta$  unchanged and letting  $r$  vary in  $\mathbb{R}_+^*$  –this is the operation of *stretching*, as defined in [BrH1]. This defines a ray in parameter space, corresponding to  $\Theta$ .

We may now consider the set  $\mathbf{Cb}$  of all possible combinatorics/critical portraits. This space is a compact finite dimensional “manifold”, endowed with a natural measure  $\mu_{\mathbf{Cb}}$  arising from the translation structure on the angle space  $\mathbb{R}/\mathbb{Z}$  (see Proposition 7.4 and Definition 7.10). As observed by [BMS], Fatou’s Theorem implies that almost every ray lands when  $r \rightarrow 0$  (Proposition 7.19). We may thus define a measurable *landing map*  $e : \mathbf{Cb} \rightarrow \mathcal{C} \subset \mathcal{P}_d$ . The basic link between critical portraits and the bifurcation measure is given by the following

**Theorem 8.** *The image of  $\mu_{\mathbf{Cb}}$  under landing is  $\mu_{\text{bif}}$ , that is,  $e_*\mu_{\mathbf{Cb}} = \mu_{\text{bif}}$ .*

The landing of external rays was extensively studied by J. Kiwi. He obtained a fundamental continuity result, which we now briefly describe. A combinatorics  $\Theta$  is of *Misiurewicz type* if all angles in  $\Theta$  are strictly preperiodic under multiplication by  $d$ . Kiwi's result [K2, Corollary 5.3] is that the landing map  $e$  is continuous at Misiurewicz parameters. This generalizes to higher degrees the well known theorem by Douady and Hubbard that rational external rays of the Mandelbrot set land.

This, combined with our description of  $\mu$  in terms of external rays, allows us to give more properties of  $\text{Supp}(\mu_{\text{bif}})$ .

**Theorem 9.** *Every Misiurewicz parameter lies inside  $\text{Supp}(\mu_{\text{bif}})$ .*

In particular the inclusion in Corollary 6 is an equality.

Finally, the landing Theorem 8 allows us to give some dynamical properties of  $\mu_{\text{bif}}$ -almost every polynomial.

**Theorem 10.** *The Topological Collet-Eckmann property holds for  $\mu_{\text{bif}}$ -almost every polynomial  $P$ .*

This result in turn implies

**Corollary 11.** *For a  $\mu_{\text{bif}}$ -generic polynomial  $P$ , we have that:*

- all cycles are repelling;
- the orbit of each critical point is dense in the Julia set;
- $K_P = J_P$  is locally connected and has Hausdorff dimension strictly less than 2.

The Topological Collet-Eckmann property (TCE for short) is a way to estimate quantitatively the recurrence of critical points. We refer to [PRS] for (many) other characterizations of the TCE condition and references. In the case of unicritical polynomials, the TCE condition is equivalent to the more standard Collet-Eckmann condition. In this case, Theorem 10 is due to Graczyk-Świątek [GŚw] and Smirnov [Sm].

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Let us close this introduction by indicating the structure of this article. In Section 1, we prove a local result, Theorem 1.1, describing the behaviour of a passive point which lands on a periodic cycle. This result is the key to the proof of Theorem 4. Sections 2 to 4 deal with general families of rational maps with a marked critical point. We begin in Section 2 with some generalities on passive and active points in families of rational maps, and briefly discuss on exceptional points. We also include the proof of Theorem 4 and a description of algebraic families of rational maps with a marked and passive critical point in the spirit of [McM2]. Section 3 is devoted to the construction of the bifurcation current. We show that its support is the activity locus. Section 4 contains the proof of a slightly more general version of our convergence result.

Sections 5 to 7 are devoted to the parameter space of polynomials. Its basic properties, as well as the parameterization by  $\mathbb{C}^{d-1}$ , are described in Section 5. In Section 6, we describe the structure of the higher bifurcation currents and prove Theorem 5 and its corollary. Section 7 is devoted to the description of  $\mu_{\text{bif}}$  in terms of external rays, leading to Theorems 8, 9 and 10 and Corollary 11.

We conclude the paper with Section 8, where we show how to extend to the space of rational maps some of the results of Section 6.

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## 1. FAMILIES OF HOLOMORPHIC GERMS AND PERIODIC POINTS

The main results of the present article deal with rational or polynomial mappings on the Riemann sphere. In the course of the proof of Theorem 1, we shall however need a result of more local nature. Our aim in this section is to describe completely the set of possible situations where a holomorphically varying point falls into a periodic cycle but does not present any bifurcation (see Theorem 1.1 below). We believe that this may have independent interest.

Before stating the theorem, let us introduce some terminology. A holomorphic family of holomorphic maps defined on the unit disk  $\Delta$  and parameterized by a complex manifold  $\Lambda$  is a holomorphic map  $f : \Lambda \times \Delta \rightarrow \mathbb{C}$ . In general, we write  $f(\lambda, z) = f_\lambda(z)$ . In this section, we always assume that  $f_\lambda(0) = 0$  for all  $\lambda$ . Notice that for fixed  $\lambda$  and  $n$ , the set of points  $U_n(\lambda) \subset \Delta$  for which  $f_\lambda, \dots, f_\lambda^n$  are well-defined forms a decreasing sequence of open neighborhoods of 0, whose intersection might be reduced to the origin.

Recall that a map  $f : \Delta \rightarrow \mathbb{C}$  fixing the origin is linearizable if there exists a holomorphic and locally invertible germ  $\phi$  such that  $\phi \circ f(z) = \mu \phi(z)$  where  $\mu = f'(0)$ . Any germ with  $|\mu| \neq 1$  is linearizable, see [CG, Mi2]. When  $\mu^l = 1$ , then  $f$  is linearizable iff  $f^l$  is the identity map. When  $|\mu| = 1$  and  $\mu$  is not a root of unity, the situation is much more complicated, but any germ is at least *formally* linearizable, see [Mi2, Problem 8.4]. The domain of linearization is the maximal *open* set  $U$  for which there exists a biholomorphism  $\phi$  into  $\mathbb{C}$  conjugating  $f$  to  $w \mapsto \mu w$ .

**Theorem 1.1.** *Let  $f_\lambda$  be any holomorphic family of holomorphic maps parameterized by a connected complex manifold  $\Lambda$ . Suppose that each  $f_\lambda$  is defined on the unit disk with values in  $\mathbb{C}$ , and leaves the origin fixed i.e.  $f_\lambda(0) = 0$ . Let  $\lambda \mapsto p(\lambda)$  be any holomorphic map such that  $p(\lambda_0) = 0$  for some parameter  $\lambda_0$ .*

*Assume that for all  $n \in \mathbb{N}$ , the function  $f_\lambda^n p(\lambda)$  is well-defined and takes its values in the unit disk. Then one of the following three cases holds.*

- (1) *For every  $\lambda \in \Lambda$ , the point 0 is attracting or superattracting, and  $p(\lambda)$  lies in the (immediate) basin of attraction of 0.*
- (2) *The point  $p$  is periodic for all parameters, i.e.  $f_\lambda^l p(\lambda) = p(\lambda)$  for some  $l$  and all  $\lambda \in \Lambda$ .*
- (3) *The multiplier of  $f_\lambda$  at 0 is constant and equals  $\exp(2i\pi\theta)$ , with  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ . For all  $\lambda \in \Lambda$ , the map  $f_\lambda$  is linearizable and  $p(\lambda)$  lies in the interior of the domain of linearization of  $f_\lambda$ .*

*Proof.* Suppose first that 0 is an attracting fixed point of  $f_{\lambda_0}$ . Our aim is to show that for all parameters, the fixed point 0 remains attracting, and  $p(\lambda)$  is attracted towards 0. Note first that there exists a neighborhood  $U$  of  $0 \in \Lambda$ , and a fixed disk  $D$  containing 0 in the dynamical plane such that  $f_\lambda(D)$  is relatively compact in  $D$  for all  $\lambda \in U$ . In particular,  $f_\lambda^n(z) \rightarrow 0$  for all  $z \in D$ .

Now by assumption the sequence  $\{f_\lambda^n p(\lambda)\}_{n \in \mathbb{N}}$  forms a normal family. Any cluster value of this sequence vanishes identically on the open set of parameters in  $U$  for which  $p(\lambda) \in D$ . So  $f_\lambda^n p(\lambda)$  actually converges to zero uniformly on compact sets on  $\Lambda$ . We infer that  $|f'_\lambda(0)| \leq 1$  for all  $\lambda$ . As the multiplier of  $f_{\lambda_0}$  at 0 has modulus  $< 1$  and  $\lambda \mapsto f'_\lambda(0)$  is holomorphic, the Maximum Principle implies that 0 is an attracting fixed point for  $f_\lambda$  for every  $\lambda$ . This shows that Case (1) holds.

From now on, we assume that the multiplier of  $f_{\lambda_0}$  at 0 has modulus  $\geq 1$ . For the sake of simplicity, we also assume that  $\Lambda$  has dimension one. We explain at the end of the proof how to deal with the general case. The key computation is contained in the next lemma. It is a classical result in the case  $\mu = 1$ , and serves as a basis for the definition of the iterative logarithm, see [Ec].

**Lemma 1.2.** *Write  $f_\lambda(z) = \mu z + \sum_{k+l \geq 2} a_{kl} \lambda^k z^l$  with  $\mu \in \mathbb{C}^*$  and  $a_{kl} \in \mathbb{C}$ . Then for any integer  $n$ , we have  $f_\lambda^n(z) = \mu^n z + \sum_{k+l \geq 2} a_{kl}^n \lambda^k z^l$ , with*

$$(1) \quad a_{kl}^n = \sum_{r=0}^l \mu^{rn} P_{rkl}(n)$$

where  $P_{rkl}(n)$  is a polynomial in  $n$ .

The proof of this lemma will be given at the end of this section. We now continue with the proof of the theorem.

For each  $n$ , write the expansion of  $\lambda$  of  $f_\lambda^n p(\lambda)$  into increasing powers as  $f_\lambda^n p(\lambda) = \sum_{j \geq 1} q_j^n \lambda^j$  with  $q_j^n \in \mathbb{C}$ . By the preceding lemma,  $f_\lambda^n p(\lambda)$  equals  $\mu^n p(\lambda) + \sum_{k+l \geq 2} a_{kl}^n \lambda^k (p(\lambda))^l$ . Identifying the  $\lambda^j$  terms of both expressions, we infer that  $q_j^n = c \cdot \mu^n + \sum_{k,l \leq j} c_{kl} a_{kl}^n$  for some constants  $c, c_{kl}$ . The order of vanishing  $\text{ord}_0(p)$  of  $p$  at 0 is greater than 1, hence  $\text{ord}_0(p^l) \geq l$ . This explains why we can take  $l \leq j$  in the above sum. In particular, for any  $j$ , we can write

$$(2) \quad q_j^n = \sum_{r=0}^j \mu^{rn} Q_{rj}(n),$$

where the  $Q_{rj}$  are polynomials.

We now translate our main assumption on  $p$  into estimates on the coefficients of the power series expansion of  $f_\lambda^n p(\lambda)$ . By assumption  $f_\lambda^n p(\lambda)$  is a family of holomorphic functions with values in the unit disk. So the Cauchy estimates imply that for each  $j$

$$(3) \quad \sup_n |q_j^n| \leq C(j) < +\infty.$$

The proof of the theorem is now based on the comparison between (2) and the estimates (3). We proceed by a case by case analysis, assuming first that  $|\mu| > 1$ ; then that  $\mu$  is a root of unity; and finally dealing with the case of  $\mu = \exp(2i\pi\theta)$ , with  $\theta$  irrational.

Suppose first  $|\mu| > 1$  (this case is classical). Fix any integer  $j \geq 1$ . Then  $q_j^n \sim c \cdot n^d \mu^{rn}$  for some  $c \in \mathbb{C}^*$  and for  $r \in \mathbb{N}$  maximal such that  $Q_{rj}$  is non-zero and  $d = \deg(Q_{rj})$ . The estimates (3) imply  $d = 0$  and  $r = 0$ , hence  $q_j^n$  is constant independent on  $n$ . We conclude that  $f_\lambda^n p(\lambda) = p(\lambda)$  for all  $n$ . In particular,  $f_\lambda p(\lambda) = p(\lambda)$ , and Case (2) of the theorem holds. Notice that since 0 is a simple root of the equation  $f_{\lambda_0}(z) - z = 0$ , in fact  $p(\lambda) \equiv 0$ .

For the remaining part of the proof, we assume  $|\mu| = 1$ , and write  $\mu = \exp(2i\pi\theta)$  for some real number  $\theta$ .

When  $\theta$  is a rational number,  $\mu$  is a root of unity and  $\mu^l = 1$  for some integer  $l$ . Equation (2) implies that  $q_j^{ln}$  is actually *polynomial* in  $n$ . The estimates (3) now show that for each  $j$  this polynomial is constant. We thus conclude that  $f_\lambda^{nl} p(\lambda) = \sum q_j \lambda^j$  for all  $n \geq 0$ . In particular,  $f^l p(\lambda) = p(\lambda)$ . As before we are in Case (2) of the theorem. Notice however that in general  $p$  need not be fixed, and that even if it is fixed, it may be different from 0.

For later reference, let us summarize this discussion in a lemma.

**Lemma 1.3.** *If the multiplier of  $f_{\lambda_0}$  at 0 has modulus  $> 1$  or is a root of unity, then  $f_{\lambda}^l p(\lambda) = p(\lambda)$  for some integer  $l$  and all  $\lambda$ .*

Finally, suppose that  $\theta$  is an irrational number, and fix some integer  $j \geq 1$ . Let  $d = \max_r \deg(Q_{rj})$ . Assume that  $d > 0$  and write  $Q_{rj}(T) = Q_{rj}^d T^d + O(T^{d-1})$ , so at least one of the  $Q_{rj}^d$  is non zero when  $r$  ranges from 0 to  $j$ . Now for any complex number  $\zeta$  of norm 1, there exists an increasing sequence of integers  $n_k$  such that  $\mu^{n_k}$  converges towards  $\zeta$ . Fix  $\zeta$  such that  $\sum_0^j \zeta^r Q_{rj}^d \neq 0$ . We infer that when  $k \rightarrow \infty$ ,  $q_j^{n_k} \sim n_k^d \times \sum_0^j \zeta^r Q_{rj}^d$ , which contradicts (3). We thus conclude that  $d = 0$ . In other words, for every  $j$  there exists a polynomial  $Q_j$  such that

$$q_j^n = Q_j(\mu^n).$$

When the point  $p$  is fixed for all parameters, Case (2) of the theorem holds so there is nothing to prove. We thus assume that outside some discrete subset of  $\Lambda$ ,  $f_{\lambda} p(\lambda) \neq p(\lambda)$  (recall that  $\dim \Lambda = 1$ ). Our aim is to prove that we are in Case (3) of the theorem. We first prove the result in the neighborhood of  $\lambda_0$ .

**Lemma 1.4.** *There exists a neighborhood  $U$  of  $\lambda_0$  such that for all  $\lambda \in U$ ,  $f_{\lambda}$  is linearizable at 0, with multiplier  $\mu = \exp(2i\pi\theta)$  independent of  $\lambda$  and  $p$  belongs to the interior of the domain of linearization. Moreover,  $f_{\lambda}^n p(\lambda) \neq f_{\lambda}^m p(\lambda)$  for all  $n \neq m$  and all  $U \ni \lambda \neq \lambda_0$ .*

We then globalize this result. First, it is clear that the multiplier of  $f_{\lambda}$  at 0 is constant equal to  $\mu$  for all  $\lambda \in \Lambda$ . Next we have:

**Lemma 1.5.** *The set of points for which  $f_{\lambda}^n p(\lambda) = f_{\lambda}^m p(\lambda)$  for some  $n > m$  is discrete in  $\Lambda$ .*

Let  $\mathcal{F}$  be the set of points for which  $p(\lambda)$  is preperiodic. The previous lemma shows it is discrete. On  $\Lambda \setminus \mathcal{F}$ , the points  $f_{\lambda}^n p(\lambda)$ ,  $n \in \mathbb{N}$ , move holomorphically and without collision, thus there is a holomorphic motion of the orbit of  $p(\lambda)$  parameterized by  $\Lambda \setminus \mathcal{F}$ . By [MSS], it automatically extends to the closure of the orbit, that we denote by  $\gamma_{\lambda}$ . For  $\lambda \notin \mathcal{F}$  close enough to  $\lambda_0$ ,  $\gamma_{\lambda}$  is a circle surrounding 0 by Lemma 1.4. As  $\Lambda \setminus \mathcal{F}$  is connected, we infer that for all  $\lambda \in \Lambda \setminus \mathcal{F}$ , the map  $f_{\lambda}$  admits an invariant quasicircle surrounding 0, on which the dynamics is conjugate to an irrational rotation. Because of the latter property, we further deduce that this quasicircle does not contain 0.

We are now in position to prove that the fixed point 0 is linearizable for any  $\lambda \in \Lambda$ . If  $\lambda \in \Lambda \setminus \mathcal{F}$ , the connected component of  $\Delta \setminus \gamma_{\lambda}$  containing 0 is a neighborhood of 0 that does not escape under iteration, hence 0 is a linearizable fixed point. If on the other hand  $\lambda \in \mathcal{F}$ , take a small loop  $\ell$  around  $\lambda$ , and avoiding  $\mathcal{F}$ . For the parameters  $\lambda' \in \ell$ , we deduce by continuity the existence of a disk  $D(0, r)$  of a fixed size in dynamical space that does not escape under iteration. By the maximum principle, this also holds true for  $\lambda$ , and we conclude that 0 is also linearizable for  $f_{\lambda}$ .

We summarize this discussion in the following lemma.

**Lemma 1.6.** *The map  $f_{\lambda}$  is linearizable for all parameters, and the point  $p(\lambda)$  lies in the closure of the domain of linearization of  $f_{\lambda}$ . In particular, either  $p(\lambda) = 0$  or its orbit is infinite.*

Finally we have:

**Lemma 1.7.** *For every  $\lambda$  the point  $p(\lambda)$  lies in the interior of the domain of linearization of  $f_{\lambda}$ .*

A proof is given below. This concludes the proof of Theorem 1.1 in case the parameter space has dimension one. The same proof gives the result in any dimension. Namely if the multiplier  $\mu$  of  $f_{\lambda_0}$  at 0 has norm  $< 1$  then  $f_\lambda^n p(\lambda)$  converges to an attracting fixed point for all parameters. We are thus in Case (1). When  $|\mu| > 1$ ,  $p(\lambda) = 0$  for each disk passing through  $\lambda_0$  hence everywhere. When  $\mu^l = 1$ , then  $f_\lambda^l p(\lambda) = p(\lambda)$  for all disks containing  $\lambda_0$  hence everywhere. In both situations, we are in Case (2). Finally when  $|\mu| = 1$  is not a root of unity and we are not in Case (2), then for any  $\lambda \in \Lambda$  there exists an immersed holomorphic disk  $D \subset \Lambda$  containing  $\lambda_0$  and  $\lambda$  (see for instance [FS2]). We can now apply the theorem to the family  $f_\lambda$  restricted to  $D$  to conclude that  $f_\lambda$  is linearizable and  $p(\lambda)$  is in the interior of the domain of linearization of  $f_\lambda$ .  $\square$

*Proof of Lemma 1.4.* We use an argument of “iteration in complex time” in the spirit of [Ec]. Pick a small disk  $U$  around  $\lambda_0$  and a small neighborhood  $D$  containing 0 in the dynamical plane such that the following hold:

- $f_\lambda p(\lambda) \neq p(\lambda)$  for all  $\lambda \in U \setminus \{\lambda_0\}$ ;
- the series  $f_\lambda^n p(\lambda) = \sum_{j \geq 1} q_j^n \lambda^j$  converges uniformly in  $U$  for all  $n$ ;
- the only solution to the equation  $f_\lambda(z) = z$  for  $z \in D$  is  $z = 0$ .

The Cauchy estimates imply that  $|q_j^n| \leq C^j$  for some constant  $C > 0$  and all  $j, n$ . As  $q_j^n = Q_j(\mu^n)$ , with  $Q_j$  a polynomial, by continuity we obtain that  $|Q_j(\zeta)| \leq C^j$  for all  $|\zeta| = 1$ . Then  $|Q_j(\zeta)| \leq C^j$  for  $|\zeta| \leq 1$  by the Maximum Principle. The map  $\phi_\lambda(\zeta) \mapsto \sum Q_j(\zeta) \lambda^j$  is thus analytic and continuous in  $\overline{\Delta} \times U$ , and by construction  $f_\lambda \phi_\lambda(\zeta) = \phi_\lambda(\mu \zeta)$ .

For  $\lambda \in U$ , the map  $\phi_\lambda$  thus semiconjugates  $f_\lambda$  to  $\zeta \mapsto \mu \zeta$  in the neighborhood of 0, but it needs not *a priori* be a conjugacy. Nevertheless we claim that for  $\lambda \in U$ ,  $f_\lambda$  is conjugate to  $\mu \zeta$  in a disk  $D(0, r)$  of fixed size. The proof of Lemma 1.7 below will actually show that the conjugacy is global.

Indeed, suppose that  $\phi_\lambda(\zeta) = 0$  for some  $\zeta \in \partial\Delta$ . Then  $\phi_\lambda(\mu^n \zeta) = 0$  for all  $n$ , hence  $\phi_\lambda \equiv 0$  on  $\partial\Delta$ , hence on  $\Delta$ . In particular  $\phi_\lambda(1) = p(\lambda) = \phi_\lambda(\mu) = f_\lambda p(\lambda) = 0$  so by assumption  $\lambda = \lambda_0$ . In particular we obtain that for  $\lambda \in \partial U$ ,  $\phi_\lambda(S^1)$  remains at definite distance  $r$  from the origin, and from this we easily deduce that for  $\lambda \in \partial U$ ,  $\phi_\lambda(\Delta) \supset D(0, r)$ .

Thanks to the semiconjugacy, for every  $\lambda \in \partial U$ , points in  $D(0, r)$  never escape under iteration. By the Maximum Principle, the same holds for  $\lambda \in U$ . From this we deduce that  $|f'_\lambda(0)| \leq 1$ . The map  $\lambda \mapsto f'_\lambda(0)$  being holomorphic, with values of modulus 1 at  $\lambda = \lambda_0$ , we conclude that it is constant equal to  $\mu$ . Any indifferent point in the Fatou set is linearizable, see [CG, Theorem II.6.2]. The sequence  $\{f_\lambda^n\}$  forms a normal family on  $D(0, r)$  for each  $\lambda \in U$ , so  $D(0, r)$  is contained in the domain of linearization. As  $p(\lambda) \rightarrow 0$  when  $\lambda \rightarrow \lambda_0$ , this implies that  $p$  is in the interior of the domain of linearization of  $f_\lambda$  for small enough  $\lambda$ .

Finally note that by construction  $f_\lambda^n p(\lambda) \neq f_\lambda^m p(\lambda)$  for all  $n \neq m$  and  $\lambda \in U \setminus \lambda_0$ .  $\square$

*Proof of Lemma 1.5.* Take a parameter  $\lambda_1$  such that  $f_{\lambda_1}^{n_1+k_1} p(\lambda_1) = f_{\lambda_1}^{k_1} p(\lambda_1)$  for some integers  $n_1, k_1 \geq 1$ . Define  $g_\lambda = f_\lambda^{n_1}$  and  $q(\lambda) = f_\lambda^{k_1} p(\lambda)$ . The point  $q(\lambda_1)$  is now fixed by  $g_{\lambda_1}$ . Let  $\mu_1$  be the multiplier of  $g_{\lambda_1}$  at  $q(\lambda_1)$ . If  $|\mu_1| < 1$ , then  $q(\lambda)$  is attracted to an attracting fixed point for all parameters which is impossible as 0 is indifferent for  $f_{\lambda_0}$ . If  $|\mu_1| > 1$  or if  $\mu_1$  is a root of unity, Lemma 1.3 implies that  $q(\lambda)$  is periodic in a neighborhood of  $\lambda_1$ , whence  $p(\lambda)$  is preperiodic for all parameters. Again this is impossible. We may thus apply Lemma 1.4 to  $g$  and  $q$ . We conclude that in a punctured neighborhood of  $\lambda_1$ , the point  $q(\lambda)$  is not  $g$ -preperiodic. This implies that  $p$  is not  $f$ -preperiodic in this neighborhood.  $\square$

*Proof of Lemma 1.7.* Recall that in the proof of lemma 1.4 we have constructed a natural holomorphically varying semiconjugacy  $\phi_\lambda(\zeta)$  between  $f_\lambda$  and  $\zeta \mapsto \mu\zeta$ . We now prove that it is a conjugacy. Indeed there exists a linearizing coordinate around zero in dynamical space, and for small  $\lambda$ ,  $p(\lambda)$  is inside the linearization domain. We denote by  $\psi_\lambda$  the unique mapping such that  $f_\lambda \circ \psi_\lambda = \psi_\lambda(\mu \cdot)$  and  $\psi'_\lambda(0) = 1$ . Let also  $s = \psi_\lambda^{-1}(z)$ , so that  $\tilde{f}_\lambda(s) = \mu s$ , where  $\tilde{f}_\lambda = \psi_\lambda^{-1} \circ f_\lambda \circ \psi_\lambda$ . For small  $\lambda$ , in the linearizing coordinate,  $\tilde{f}_\lambda^n(\psi_\lambda^{-1} p(\lambda)) = \mu^n \psi_\lambda^{-1} p(\lambda)$ , so that by definition of  $\phi_\lambda$  we have  $\psi_\lambda^{-1} \circ \phi_\lambda(\zeta) = \zeta \psi_\lambda^{-1} p(\lambda)$ . In particular,  $\phi_\lambda$  is a conjugacy for small  $\lambda$ , and

$$(4) \quad \phi_\lambda(\zeta) = \psi_\lambda(\zeta \psi_\lambda^{-1} p(\lambda))$$

Let us now consider the radius of convergence  $R(\lambda)$  of the power series defining  $\zeta \mapsto \phi_\lambda(\zeta)$ . We know that this function is defined on the unit disk so  $R(\lambda) \geq 1$ . Since  $\phi_\lambda(1) = p(\lambda)$ , to get the desired conclusion it is enough to prove that  $R > 1$  everywhere. This will be a consequence of a subharmonicity property of  $R$ .

For  $\lambda$  close to zero,  $\phi_\lambda$  is  $p(\lambda)$  multiplied by the linearizing coordinate so  $R(\lambda) > 1$  since  $p(\lambda)$  is inside the linearization domain. Write  $\phi_\lambda(\zeta) = \sum_{k \geq 0} a_k(\lambda) \zeta^k$ . The coefficient  $a_k(\lambda)$  is defined by the formula  $a_k(\lambda) = \int_{S^1} \phi_\lambda(\zeta) \zeta^{-k}$  so it depends holomorphically on  $\lambda$ . Since  $\phi_\lambda$  takes its values in the unit disk,  $|a_k| \leq 1$ . The radius of convergence equals  $R^{-1}(\lambda) = \limsup_k |a_k(\lambda)|^{1/k}$ . As  $|a_k| \leq 1$ , the function  $-\log R$  is the supremum of a sequence of non-positive subharmonic function on  $\Lambda$ . Its upper-semicontinuous regularization,  $\rho^*$  thus defines a non-positive subharmonic function. As  $\lambda \rightarrow 0$ ,  $p(\lambda) \rightarrow 0$  and  $f_\lambda$  is linearizable on a fixed disk  $D(0, r)$ , so  $R(\lambda) \rightarrow \infty$  (see (4)), hence  $\rho^* < 0$  there. By the Maximum Principle, we conclude that  $\rho^* < 0$  everywhere. As  $-\log R \leq \rho^*$ , we get that  $R(\lambda) > 1$  for all  $\lambda \in \Lambda$ .  $\square$

*Proof of Lemma 1.2.* Write  $f_\lambda^n(z) = \mu^n z + \sum_{k+l \geq 2} a_{kl}^n \lambda^k z^l$ . Then  $f^{n+1} = f^n \circ f$  from which we deduce that

$$f_\lambda^{n+1}(z) = \mu^n \left( \mu z + \sum_{k+l \geq 2} a_{kl} \lambda^k z^l \right) + \sum_{i+j \geq 2} a_{ij}^n \lambda^i \left( \mu z + \sum_{p+q \geq 2} a_{pq} \lambda^p z^q \right)^j.$$

For  $k + l \geq 2$ , we thus infer that

$$a_{kl}^{n+1} = \mu^n a_{kl} + \sum_{i+j \geq 2} a_{ij}^n \times \text{Term in } \lambda^{k-i} z^l \text{ of } \left[ \left( \mu z + \sum_{p+q \geq 2} a_{pq} \lambda^p z^q \right)^j \right]$$

The sum over  $(i, j)$  in the right hand side is finite, as we necessarily have  $i \leq k$  and  $j \leq l$ . Note also that for  $(i, j) = (k, l)$ , the contribution of the sum is exactly  $\mu^l a_{kl}^n$ . We conclude that there exist constants  $c, c_{ij}$ , independent of  $n$ , such that

$$a_{kl}^{n+1} = \mu^l a_{kl}^n + c \cdot \mu^n + \sum_{(i,j) < (k,l), i+j \geq 2} c_{ij} a_{ij}^n,$$

where  $\leq$  (resp.  $<$ ) here denotes the partial order on  $\mathbb{N}^2$  given by  $(i, j) \leq (k, l)$  iff  $i \leq k$  and  $j \leq l$  (resp. at least one inequality is strict).

We complete the proof by an induction on  $(k, l)$  with respect to the previously defined partial order. Indeed if for every  $(i, j) < (k, l)$ ,  $a_{ij}^n$  is of the form  $\sum_{r=0}^j \mu^{rn} P_{rij}(n)$  we get that

$$a_{kl}^{n+1} = \mu^l a_{kl}^n + \sum_{r=0}^l \mu^{rn} Q_{rkl}(n).$$

Since for every complex number  $\nu$ , the sum  $\sum_{k=0}^n \nu^k k^d$  is of the form  $\nu^n P(n)$  where  $P$  is a polynomial, the sequence  $a_{kl}^n$  is of the required form (1).  $\square$

## 2. GENERALITIES ON FAMILIES OF RATIONAL MAPS

**2.1. Active and passive points.** We start with a definition.

**Definition 2.1.** A holomorphic family of rational maps of degree  $d$  with a marked critical point is a triple  $(\Lambda, f, c)$  such that:

- $\Lambda$  is a smooth connected complex manifold;
- $f : \Lambda \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is a holomorphic map;
- for any  $\lambda \in \Lambda$ , the map  $z \mapsto f_\lambda(z) := f(\lambda, z)$  is a rational map of  $\mathbb{P}^1$  of degree  $d$ ;
- $c : \Lambda \rightarrow \mathbb{P}^1$  is a holomorphic map such that  $f'_\lambda c(\lambda) = 0$  for all  $\lambda$ .

We call  $\Lambda$  the parameter space of the family.

Throughout the paper we will focus our interest on the locus of points in parameter space for which  $c(\lambda)$  is dynamically unstable. This is a classical notion, which goes back to Levin [Le1] and Lyubich [Ly2]. We use the terminology of McMullen [McM1].

**Definition 2.2.** The marked critical point  $c$  is passive at  $\lambda_0 \in \Lambda$  if  $\{\lambda \mapsto f_\lambda^n c(\lambda)\}_{n \in \mathbb{N}}$  forms a normal family of holomorphic functions in the neighborhood of  $\lambda_0$ . Otherwise  $c$  is said to be active at  $\lambda_0$ . The set of points  $\mathcal{A} \subset \Lambda$  where  $c$  is active is called the activity locus.

Notice that the family  $(f_\lambda)$  being structurally stable on its Julia set is actually equivalent to the fact that all marked critical points are passive, see e.g. [McM2]. In particular the results of [MSS] imply that the complement of the activity locus is always an open dense set in  $\Lambda$ .

**Lemma 2.3.** Let  $(\Lambda, f, c)$  be a holomorphic family of rational maps with a marked critical point. If  $c$  is active at  $\lambda_0$ , then there exists a nearby parameter  $\lambda$  such that  $c(\lambda)$  is prerepelling (i.e. preperiodic to a repelling cycle).

*Proof.* The lemma follows from a classical normal family argument. Fix three repelling periodic points at the parameter  $\lambda_0$ . They persist in a neighborhood of  $\lambda_0$ . By conjugating with a holomorphically varying Möbius transformation, we may further assume they persistently equal  $\{0, 1, \infty\}$ . Since the family  $\{f_\lambda^n c(\lambda)\}_{n \in \mathbb{N}}$  is not normal in any neighborhood of  $\lambda_0$ , it cannot avoid these three points.  $\square$

The following result is a little bit more delicate (see also [Le1, McM1]).

**Proposition 2.4.** Suppose  $c$  is active at  $\lambda_0$ . In any neighborhood of  $\lambda_0$ , there exists a parameter  $\lambda$  for which  $f_\lambda^n c(\lambda)$  converges to an attracting cycle when  $n \rightarrow \infty$ .

*Proof.* We will actually prove that there exists a parameter close to  $\lambda_0$  where  $c$  is periodic, hence superattracting. Replacing  $\Lambda$  by a suitable finite ramified cover, we may follow holomorphically a preimage of  $c(\lambda)$ . Observe that this operation preserves active and passive critical

points.. We thus get a holomorphic map  $c_{-1}(\lambda)$  such that  $f_\lambda(c_{-1}(\lambda)) = c(\lambda)$ . Similarly we get a holomorphic map  $c_{-2}$  with  $f_\lambda c_{-2}(\lambda) = c_{-1}(\lambda)$ . There are two cases: either  $c(\lambda)$ ,  $c_{-1}(\lambda)$  and  $c_{-2}(\lambda)$  are disjoint in the neighborhood of  $\lambda_0$  or there is a parameter near  $\lambda_0$  where  $c$  is fixed. In the latter case we are done. In the former, we can choose a holomorphically varying family of Möbius maps  $\phi_\lambda$  such that  $\phi_\lambda c(\lambda) = 0$ ,  $\phi_\lambda c_{-1}(\lambda) = 1$  and  $\phi_\lambda c_{-2}(\lambda) = \infty$ . Replacing  $f_\lambda$  by the family  $\phi_\lambda \circ f_\lambda \circ \phi_\lambda^{-1}$  (still denoted by  $f_\lambda$ ), we get that  $\infty \mapsto 1 \mapsto 0 \equiv c(\lambda)$ . Now if  $c$  is active at  $\lambda_0$ , the family  $(f_\lambda^n c(\lambda))$  cannot avoid  $\{0, 1, \infty\}$  and we conclude that there are parameters close to  $\lambda_0$  where  $c$  becomes periodic.  $\square$

We now prove Theorem 4 stated in the introduction, which classifies the dynamics of passive critical points. This is essentially a consequence of Theorem 1.1.

*Proof of Theorem 4.* Suppose  $c$  is not stably preperiodic, and  $f_{\lambda_0}^{n_0+k_0} c(\lambda_0) = f_{\lambda_0}^{n_0} c(\lambda_0)$  for some  $\lambda_0 \in U$ , and minimal  $n_0 \geq 0$ ,  $k_0 \geq 1$ . By Theorem 1.1, there exists a periodic cycle  $\{p(\lambda), \dots, f^{k_0-1}(p(\lambda))\}$  which is either attracting or with multiplier  $\exp(2i\pi\theta)$  with  $\theta$  irrational such that  $p(\lambda_0) = f_{\lambda_0}^{n_0} c(\lambda_0)$ ; and either  $c$  is attracted towards the orbit of  $p$  for all parameters, or  $c$  eventually lies in the interior of the domain of linearization of  $p$  (hence  $f$  has a Siegel disk at  $p$ ).

Suppose we are in the first case. Pick a small neighborhood  $U$  of  $\lambda_0$ , and small disjoint disks  $U_i$  such that

- $U_0 = U_{k_0} \ni p(\lambda), U_1 \ni f_\lambda p(\lambda), \dots, U_{k_0-1} \ni f_\lambda^{k_0-1} p(\lambda)$  for all  $\lambda \in U$ ;
- $f_\lambda$  is injective on  $U_i$  for all  $i$  and all  $\lambda$ ;
- $f_\lambda U_i$  is relatively compact in  $U_{i+1}$  for all  $i$  and all  $\lambda$ .

Replacing  $U$  by an even smaller open set containing  $\lambda_0$ , we may assume that  $f_\lambda^n c(\lambda) \in \cup_{i=0}^{k_0-1} U_i$  for all  $\lambda \in U$  and all  $n$ . It is then clear that  $c(\lambda)$  is preperiodic iff  $f_\lambda^n c(\lambda) = p(\lambda)$  for some  $n$ , and that it is equivalent to  $f_\lambda^{n_0} c(\lambda) = p(\lambda)$ . The latter condition defines a closed analytic subspace in  $U$ .

When  $p$  is neutral, the proof goes through with the following additional remarks. Replace the last condition in the construction of  $U_i$  above by taking  $U_i$  to be an open set containing the closure of the orbit of  $f_\lambda^{n_0} c(\lambda)$  for all parameters. This is possible even though the Siegel disk need not move continuously with  $\lambda$ . Indeed, by the proof of lemma 1.4, for  $\lambda$  close to  $\lambda_0$ , there is a disk of fixed size  $D(p(\lambda), r)$  contained in the linearization domain.

As before, we conclude that  $c(\lambda)$  is preperiodic iff  $f_\lambda^{n_0+k_0}(c(\lambda)) = f_\lambda^{n_0}(c(\lambda))$ .  $\square$

**2.2. Quasiprojective parameter space.** Following McMullen, call *isotrivial* a family in which any two members are conjugate by a Möbius transformation; and *algebraic* a family parameterized by a quasiprojective variety.

McMullen proved in [McM2] that any non isotrivial algebraic family of rational maps admits bifurcations, with the only exception of families of *flexible Lattès examples*. A consequence of this result is that any non isotrivial algebraic family is critically finite or has bifurcations. Here, building on [McM2], we prove the more precise result that in a non isotrivial algebraic family of rational maps *each* critical point either presents bifurcations or is stably preperiodic.

**Theorem 2.5.** *Let  $(\Lambda, f, c)$  be a holomorphic family of rational maps of degree  $d \geq 2$  with a marked critical point, where  $\Lambda$  is an irreducible quasiprojective complex variety. Assume that the activity locus of  $c$  is empty. Then:*

- either all maps  $f_\lambda$  are conjugate to each other by Möbius transformations;

- or there exist integers  $(n, k)$  such that for every  $\lambda \in \Lambda$ ,  $f_\lambda^n(c_\lambda) = f_\lambda^k(c_\lambda)$ .

*Proof.* Taking successive hyperplane sections, we reduce the proof to the case where  $\Lambda$  is a Riemann surface of finite type, that is a compact Riemann surface with finitely many points deleted. The crucial fact is that there are only finitely many non constant holomorphic maps  $\Lambda \rightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}$  (see [McM2, p.478] for a proof).

We start with an easy lemma.

**Lemma 2.6.** *Let  $(\Lambda, f, c)$  be as above, and assume that  $c$  is not stably preperiodic. Assume that for large  $n$ , the function  $f_\lambda^n c(\lambda)$  is constant and does not depend on  $\lambda$ . Then the family is constant, that is, for all  $\lambda, \lambda'$ , we have  $f_\lambda = f_{\lambda'}$ .*

*Proof.* For some integer  $n_0$  and all  $n \geq n_0$ ,  $u_n = f_\lambda^n c(\lambda)$  does not depend on  $\lambda$ . Moreover, since there exists a parameter for which  $c$  has infinite orbit, the set  $\{u_n, n \geq n_0\}$  is infinite. Take  $\lambda \neq \lambda'$ . Then  $f_\lambda(u_n) = f_{\lambda'}(u_n)$  for all  $n \geq n_0$ . Any two rational maps of constant degree, agreeing on an infinite set are equal. Hence  $f_\lambda = f_{\lambda'}$ .  $\square$

We continue with the proof of the theorem. We thus assume that  $c$  is passive throughout  $\Lambda$ , and not stably preperiodic. We shall prove that any two rational maps in this family are conjugated by a Möbius transformation.

Suppose first that  $c$  is *never* periodic. As in Proposition 2.4, replacing  $\Lambda$  by a suitable finite ramified cover, we may follow holomorphically two preimages of  $c(\lambda)$ . We thus get holomorphic maps  $c_{-1}(\lambda)$  and  $c_{-2}(\lambda)$  such that  $f_\lambda(c_{-1}(\lambda)) = c(\lambda)$  and  $f_\lambda(c_{-2}(\lambda)) = c_{-1}(\lambda)$ . Note that any finite ramified cover of a Riemann surface of finite type is of finite type too. Also by assumption,  $c$  is never periodic so  $c(\lambda)$ ,  $c_{-1}(\lambda)$  and  $c_{-2}(\lambda)$  are always distinct. As before we can thus choose a holomorphically varying family of Möbius maps  $\phi_\lambda$  such that  $\phi_\lambda c(\lambda) = 0$ ,  $\phi_\lambda c_{-1}(\lambda) = 1$  and  $\phi_\lambda c_{-2}(\lambda) = \infty$  and we replace  $f_\lambda$  by the family  $\phi_\lambda \circ f_\lambda \circ \phi_\lambda^{-1}$ .

As  $c$  is never periodic, for any  $n > 0$  and any  $\lambda \in \Lambda$ ,  $f_\lambda^n(c(\lambda))$  avoids  $\{0, 1, \infty\}$ . So the family of holomorphic maps  $f_\lambda^n c(\lambda) : \Lambda \rightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}$  takes only finitely many nonconstant terms. Since we have moreover assumed that  $f_\lambda^n(c(\lambda))$  is not persistently preperiodic,  $f_\lambda^n(c(\lambda))$  is constant for large  $n$ . By Lemma 2.6, we conclude that the family is constant.

Suppose then that  $c$  is *periodic* for some parameter. We may assume that for some  $\lambda_0$ ,  $f_{\lambda_0}^{k_0} c(\lambda_0) = c(\lambda_0)$ , with minimal  $k_0$ . Using Theorem 4, we infer that  $c$  is attracted to an attracting periodic orbit throughout  $\Lambda$ . The multiplier of this periodic orbit defines a holomorphic function on a Riemann surface of finite type with values in the unit disk. It is hence constant equal to 0.

We conclude that there is a persistent superattracting cycle  $\{p(\lambda), \dots, f_\lambda^{k_0} p(\lambda)\}$  attracting  $c$ , and  $c(\lambda_0) = p(\lambda_0)$ . The cycle can be followed holomorphically because the multiplier never equals 1. Replacing the family  $\{f_\lambda\}$  by  $\{f_\lambda^{k_0}\}$ , we may assume that  $p(\lambda)$  is a fixed point. Let  $m - 1$  be the largest integer  $q$  such that  $\frac{d^q f}{d\lambda^q} p(\lambda)$  is identically zero. Thanks to the Böttcher theorem, outside some discrete subset  $\mathcal{D} \subset \Lambda$ ,  $f_\lambda(\zeta) = \zeta^m$  in a suitable coordinate  $\zeta$  centered at  $p(\lambda)$ . In this case, we can even choose  $\zeta$  to depend holomorphically on  $\lambda$ . On the complement  $\mathcal{D}$ , the multiplicity is larger so  $f_\lambda(\zeta) = \zeta^{m_\lambda}$  for some integer  $m_\lambda > m$ .

Let  $d_{\mathbb{P}^1}$  be the spherical distance on  $\mathbb{P}^1$  ( $d_{\mathbb{P}^1} \leq 1$ ). If  $z$  lies in the basin of attraction of  $p(\lambda)$ , let

$$g_\lambda(z) = \lim_{n \rightarrow \infty} \frac{1}{m^n} \log d_{\mathbb{P}^1}(f_\lambda^n(z), p(\lambda)).$$

Notice that the limit is the same when one replaces  $d_{\mathbb{P}^1}$  by any equivalent distance. In the local chart  $\zeta$  mentioned above, we thus get  $g_\lambda(\zeta) = \log |\zeta|$  for any  $\lambda \notin \mathcal{D}$ . It follows that  $(z, \lambda) \mapsto g_\lambda(z)$  is plurisubharmonic and continuous in the domain consisting of couples  $(z, \lambda)$  such that  $\lambda \notin \mathcal{D}$  and  $z$  in the basin of  $p$ .

In particular the function  $\lambda \mapsto g_\lambda c(\lambda)$  is subharmonic, non positive, and continuous outside the discrete subset of  $\mathcal{D} \subset \Lambda$ . It may thus be extended across  $\mathcal{D}$  as a non positive plurisubharmonic function on  $\Lambda$ . Now  $\Lambda$  is a Riemann surface of finite type, hence  $g_\lambda c(\lambda)$  extends across the punctures of  $\Lambda$  and defines a non positive subharmonic function a *compact* Riemann surface. It is thus constant. Since  $g_{\lambda_0} c(\lambda_0) = -\infty$ , we get that  $g_\lambda c(\lambda) \equiv -\infty$ , that is  $c(\lambda) \equiv p(\lambda)$  is persistently periodic.

This concludes the proof of the theorem.  $\square$

**2.3. Exceptional points.** To get a precise statement of our convergence theorem, we need a short discussion on exceptional points. Let  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be any rational map of degree  $d \geq 2$ . A point  $z$  in the Riemann sphere is called *exceptional*, if  $z$  is totally invariant by  $f$  or by  $f^2$ . The set of exceptional points of  $f$  is denoted by  $\mathcal{E}(f)$ , and contains at most two points. When  $\text{card}(\mathcal{E}(f)) = 1$ , then  $f$  is conjugate to a polynomial for which  $\mathcal{E}(f) = \{\infty\}$ . When  $\text{card}(\mathcal{E}(f)) = 2$ , then  $f$  is conjugate to  $z \mapsto z^d$  for some integer  $d \in \mathbb{Z} \setminus \{\pm 1, 0\}$ .

Let now  $(\Lambda, f)$  be a holomorphic family of rational maps. Define  $\mathcal{E} := \{(\lambda, z), z \in \mathcal{E}(f_\lambda)\}$ . This defines a closed analytic subset in  $\Lambda \times \mathbb{P}^1$ . It can be empty, or reducible, or singular. For the sake of simplicity, we write  $e$  for the cardinality of  $\mathcal{E}(f_\lambda)$  for generic  $\lambda$ . As  $\Lambda$  is connected, this is precisely  $\min_\Lambda \text{card}(\mathcal{E}(f_\lambda))$ .

$e = 2$ . Then  $\text{card}(\mathcal{E}(f_\lambda)) = 2$  for all  $\lambda$ . In this case,  $\pi_1$  induces a non-ramified 2 to 1 cover of  $\mathcal{E}$  onto  $\Lambda$ , and  $\mathcal{E}$  is a smooth hypersurface. Locally at any parameter  $\lambda_0$ , the family is conjugate to the trivial family  $f_\lambda(z) = z^d$ ,  $d \in \mathbb{Z} \setminus \{\pm 1, 0\}$ . In general, the family need not be globally trivial (e.g.  $f_\lambda(z) = \lambda z^d$  over  $\mathbb{C}^*$ ).

$e = 1$ . At a generic point,  $\text{card}(\mathcal{E}(f_\lambda)) = 1$ . Let  $\mathcal{E}'$  be the irreducible component of  $\mathcal{E}$  containing  $\mathcal{E}(f_\lambda)$  for generic  $\lambda$ . Then  $\pi_1 : \mathcal{E}' \rightarrow \Lambda$  is an isomorphism, and after passing to the universal cover of  $\Lambda$ , we may assume that  $\pi_2(\mathcal{E}') = \infty \in \mathbb{P}^1(\mathbb{C})$ . In particular,  $f_\lambda$  is a *polynomial* for all  $\lambda$ .

$e = 0$ . This is the generic case. Outside a proper closed Zariski subset of  $\Lambda$ , the set  $\mathcal{E}(f_\lambda)$  is *empty*.

### 3. THE BIFURCATION CURRENT

We now associate a natural positive closed  $(1, 1)$  current to the data  $(\Lambda, f, c)$ . This will be the main object of interest in the paper. We refer the reader to Demainly [De] for basics on positive closed currents.

We first fix some notation. Write  $\widehat{\Lambda} := \Lambda \times \mathbb{P}^1$ . The family of maps  $f_\lambda$  lifts to a holomorphic map  $\widehat{f} : \widehat{\Lambda} \rightarrow \widehat{\Lambda}$  sending  $(\lambda, z)$  to  $(\lambda, f_\lambda(z))$ . We denote by  $\pi_1 : \widehat{\Lambda} \rightarrow \Lambda$  and  $\pi_2 : \widehat{\Lambda} \rightarrow \mathbb{P}^1$  the two natural projections. The map  $\widehat{p} : \Lambda \rightarrow \widehat{\Lambda}$  defined by  $\widehat{p}(\lambda) = (\lambda, p(\lambda))$  induces an isomorphism from  $\Lambda$  onto its image  $\Gamma$  which is a smooth submanifold of  $\widehat{\Lambda}$ . It defines a section of  $\pi_1$ , that is,  $\pi_1 \circ \widehat{p}$  is the identity map.

Let now  $\omega$  be the unique smooth positive closed  $(1, 1)$  form on  $\mathbb{P}^1$  which is invariant under the unitary group, and normalized by  $\int_{\mathbb{P}^1} \omega = 1$  (the Fubini-Study form). It induces the spherical distance on  $\mathbb{P}^1$ , which we denote by  $d_{\mathbb{P}^1}$ . We set  $\widehat{\omega} := \pi_2^* \omega$ .

**Proposition-Definition 3.1.** *For any holomorphic family  $(\Lambda, f, c)$  of rational maps of degree  $d \geq 2$  with a marked critical point, the sequence of positive closed  $(1, 1)$  currents  $d^{-n} \hat{f}^{n*} \hat{\omega}$  converges to a positive closed  $(1, 1)$  current  $\widehat{T}$  on  $\widehat{\Lambda}$ .*

*This current admits a continuous potential locally at any point. The image under  $\pi_1$  of the restriction of  $\widehat{T}$  to the submanifold  $\Gamma := \{(\lambda, c(\lambda)) \in \widehat{\Lambda}\}$  is thus well-defined as a positive closed  $(1, 1)$  current on  $\Lambda$ . It is called the bifurcation current of the family, and we denote it by  $T$ .*

*Proof.* All assertions are *local* in  $\Lambda$ . We may thus assume that  $\Lambda$  is the unit open polydisk in  $\mathbb{C}^n$ , and that the family is defined over a neighborhood of the closed unit polydisk.

We first claim the existence of a continuous and bounded function  $\hat{g}$  on  $\widehat{\Lambda}$  such that

$$(5) \quad d^{-1} \hat{f}^* \hat{\omega} - \hat{\omega} = dd^c \hat{g}.$$

To see this, choose homogeneous coordinates  $[z : w]$  on  $\mathbb{P}^1$ , and let  $\pi : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$  be the natural projection. Take a lift of the family of rational maps  $f$  to  $\mathbb{C}^2$ . Write  $f_\lambda[z : w] = [P_\lambda(z, w) : Q_\lambda(z, w)]$  where  $P_\lambda, Q_\lambda$  are two holomorphic families of homogeneous polynomials of degree  $d$  with no common factors (for each  $\lambda$ ). This is always possible by restricting  $\Lambda$  if necessary. The function

$$\hat{g}(\lambda, z) := \frac{1}{2d} \log \frac{|P_\lambda(z, w)|^2 + |Q_\lambda(z, w)|^2}{(|z|^2 + |w|^2)^d}$$

clearly satisfies our requirements.

Applying  $d^{-k} f^{k*}$  to (5), and summing from  $k = 1$  to  $n - 1$ , we get that  $d^{-n} \hat{f}^{n*} \hat{\omega} - \hat{\omega} = dd^c \sum_0^{n-1} d^{-k} \hat{g} \circ \hat{f}^k$ . As  $\sup_{\widehat{\Lambda}} |d^{-k} \hat{g} \circ \hat{f}^k| \leq d^{-k} \sup_{\widehat{\Lambda}} |\hat{g}|$ , the sequence of functions  $\sum_0^{n-1} d^{-k} \hat{g} \circ \hat{f}^k$  converges uniformly on  $\widehat{\Lambda}$  to a continuous function  $\hat{g}_\infty$ . In particular,  $d^{-n} \hat{f}^{n*} \hat{\omega} \rightarrow \widehat{T} := \hat{\omega} + dd^c \hat{g}_\infty$ .

The restriction to a subvariety of a continuous  $(1, 1)$  form and of a continuous function is always well-defined. We may thus set  $\widehat{T}|_\Gamma := \hat{\omega}|_\Gamma + dd^c(\hat{g}_\infty|_\Gamma)$ . Finally,  $\pi_1$  is an isomorphism from  $\Gamma$  onto  $\Lambda$ , so we may define  $(\pi_1)_*(\widehat{T}|_\Gamma)$  as the bifurcation current.  $\square$

**Theorem 3.2.** *The support of the bifurcation current coincides with the activity locus.*

From this theorem and the convergence statement in Definition 3.1, we get the following interesting corollary.

**Corollary 3.3.** *Assume that there exists an increasing sequence of integers  $(k_n)$  such that  $\{f_\lambda^{k_n} c(\lambda)\}$  is a normal family in some neighborhood of  $\lambda_0$ . Then  $c$  is passive at  $\lambda_0$ .*

*Proof of Theorem 3.2.* Suppose first that  $c$  is passive at  $\lambda_0$ . We may thus assume that  $f_\lambda^{k(n)} c(\lambda) \rightarrow h(\lambda)$  in a neighborhood  $U$  of  $\lambda_0$  for some sequence  $k(n)$  increasing to infinity. Notice that the map  $\lambda \rightarrow f_\lambda^n c(\lambda)$  is identical to the composite map  $\pi_1 \circ \hat{f}^n \circ \hat{p}$ . On  $U$ , we infer that  $(\pi_1 \circ \hat{f}^{k(n)} \circ \hat{p})^* \omega - h^* \omega$  tends to zero in the supremum norm of forms (hence in the weak topology of currents). By construction, we have  $T|_U = \lim_{n \rightarrow \infty} d^{-n} (\pi_1 \circ \hat{f}^n \circ \hat{p})^* \omega$ , whence  $T|_U \equiv 0$ .

Conversely, assume that  $T|_U \equiv 0$  in a neighborhood  $U$  of  $\lambda_0$ . We need to show that the sequence  $\{f_\lambda^n c(\lambda)\}$  is normal in a neighborhood of  $\lambda_0$ . We may assume without loss of generality that  $\dim(\Lambda) = 1$ .

On  $U$ , we have

$$(\pi_1 \circ \hat{f}^n \circ \hat{p})^* \omega = d^n \left( d^{-n} (\pi_1 \circ \hat{f}^n \circ \hat{p})^* \omega - T \right) = dd^c \left[ d^n \sum_n^\infty \frac{\hat{g} \circ \hat{f}^k \circ \hat{p}}{d^k} \right],$$

where  $dd^c \hat{g} = d^{-1} \hat{f}^* \hat{\omega} - \hat{\omega}$  as in the previous proof. The function  $\hat{g}$  being continuous, the sequence  $d^n \sum_n^\infty d^{-k} \hat{g} \circ \hat{f}^k \circ \hat{p}$  is uniformly bounded in  $n$ . In particular,  $(\pi_1 \circ \hat{f}^n \circ \hat{p})^* \omega = dd^c \phi_n$  for a sequence of continuous functions  $\phi_n$  with  $|\phi_n| \leq C$ . Note that  $dd^c \phi_n$  is positive, thus  $\phi_n$  is psh on  $U$ .

Let  $\omega_\Lambda$  be any smooth positive  $(1, 1)$  form on  $U$ . As  $\dim(\Lambda) = 1$ , we have

$$(\pi_1 \circ \hat{f}^n \circ \hat{p})^* \omega = \|d(f_\lambda^n c(\lambda))\|^2 \omega_\Lambda,$$

where  $\|d(f_\lambda^n c(\lambda))\|$  is the norm of the differential of the map  $\lambda \mapsto f_\lambda^n c(\lambda)$  computed in the metrics induced by  $\omega_\Lambda$  in  $\Lambda$  and  $\omega$  in  $\mathbb{P}^1$ . Fix any relatively compact open set  $V \subset\subset U$ . The relative capacity of the compact set  $\overline{V}$  with respect to  $U$  is by definition the non-negative real number  $\text{cap}_U(\overline{V}) := \sup\{\int_{\overline{V}} dd^c u, 0 \leq u \leq 1, u \text{ psh}\}$ , see [BT1]. In particular, we obtain

$$\int_V \|d(f_\lambda^n c(\lambda))\|^2 \omega_\Lambda \leq \int_{\overline{V}} dd^c(\phi_n + C) \leq 2C \text{cap}_U(\overline{V}).$$

When  $V$  decreases to a point, its capacity tends to zero. In particular, by choosing  $V$  sufficiently small, we may assume that  $\sup_n \int_V \|d(f_\lambda^n c(\lambda))\|^2 \omega_\Lambda < \varepsilon_0 < 1$ .

Now look at the sequence of (closed) analytic curves  $\Gamma_n = \{(\lambda, \hat{f}^n c(\lambda))\} \subset V \times \mathbb{P}^1$ . The area of  $\Gamma_n$  with respect to the metric  $\pi_1^* \omega_\Lambda + \pi_2^* \omega$  is precisely  $\int_V (1 + \|d(f_\lambda^n c(\lambda))\|^2) \omega_\Lambda$ . Reducing  $V$  again if necessary, we may assume that  $\sup_n \text{Area}(\Gamma_n) < \varepsilon_0 < 1$ . By Bishop's theorem, from any subsequence, one can extract a (sub-)subsequence converging in the Hausdorff topology to an analytic curve  $\Gamma_\infty$ . The area of  $\Gamma_\infty$  is also less than  $\varepsilon_0 < 1$ , thus  $\Gamma_\infty$  cannot contain any fiber of  $\pi_2$  (the area of any such fiber is  $\text{Area}_\omega(\mathbb{P}^1) = 1$ ). All curves  $\Gamma_n$  being graphs over  $\Lambda$ , we conclude that the limit  $\Gamma_\infty$  itself is a graph over  $\Lambda$ .

We have thus shown that on a fixed small neighborhood  $V$  of  $\lambda_0$ , any subsequence of  $\{f_\lambda^n c(\lambda)\}$  admits a converging subsequence, since the associated sequence of graphs does. We conclude that  $\{f_\lambda^n c(\lambda)\}$  is a normal family on  $V$ . Hence  $c$  is passive at  $\lambda_0$ .  $\square$

#### 4. THE CONVERGENCE THEOREM

We now come to the proof of Theorem 1.

**Definition 4.1.** Let  $(\Lambda, f, c)$  be a holomorphic family of rational maps of  $\mathbb{P}^1$  with one marked critical point. For any  $n \neq m \geq 0$ , we denote by  $\text{Per}(n, m)$  the (not necessarily reduced) analytic subset of  $\Lambda$  defined by the equation  $f_\lambda^n c(\lambda) = f_\lambda^m c(\lambda)$ . For the sake of simplicity, we write  $\text{Per}(n)$  for  $\text{Per}(n, 0)$ . As a set, it consists of the parameters for which  $c(\lambda)$  is periodic of period dividing  $n$ .

A family is called trivial when  $\text{Per}(n, m) = \Lambda$  for some  $n \neq m$ .

We stress that we consider the analytic sets  $\text{Per}(n, m)$  as defined by the equations  $f_\lambda^n c(\lambda) = f_\lambda^m c(\lambda)$ , so they can come along with multiplicities. Note that the activity locus is empty in the case where  $\Lambda$  is a trivial family. Note also that when the cardinality of the exceptional set of a generic map is 2, then the family is trivial.

In the sequel, we shall work exclusively with non-trivial families of rational maps of  $\mathbb{P}^1$ . For any such family and any  $n \neq m$ ,  $\text{Per}(n, m)$  is a hypersurface in  $\Lambda$ , which might be singular, or reducible, or even non-reduced. For convenience, we state again the convergence theorem.

**Theorem 4.2.** *Let  $(\Lambda, f, c)$  be a non-trivial holomorphic family of rational maps of  $\mathbb{P}^1$  of degree  $d \geq 2$  with a marked critical point. We moreover assume one of the following two assumptions holds.*

- (H1) *The manifold  $\Lambda$  is quasi-projective.*
- (H2) *For any  $\lambda \in \Lambda$ , there exists an immersed analytic curve  $\Gamma \subset \Lambda$  containing  $\lambda$  such that the closure of the complement of  $\{\lambda, c(\lambda)\}$  is attracted to a periodic cycle} is a proper compact subset of  $\Gamma$ .*

Let also  $e \in \{0, 1\}$  be the cardinality of the exceptional set of a generic  $f_\lambda$ . Then for any sequence of integers  $0 \leq k(n) < n$ , the following convergence statement holds:

$$\lim_{n \rightarrow \infty} \frac{[\text{Per}(n, k(n))]}{d^n + d^{(1-e)k(n)}} = T$$

Notice that in the assumption (H2), we allow the curve  $\Gamma$  to be singular, or non properly embedded in  $\Lambda$ . Recall also that we have classified in Theorem 2.5 the algebraic families of rational maps for which  $T \equiv 0$ . We refer to the next sections for applications of this result to concrete families.

An important fact here is that the subvarieties  $\text{Per}(n)$ ,  $\text{Per}(n, k)$ , have several irreducible components: e.g. if  $d|n$ ,  $\text{Per}(d) \subset \text{Per}(n)$ . Also periodic points are preperiodic so  $\text{Per}(n - k) \subset \text{Per}(n, k)$ . We a priori have no control on the multiplicities of the various irreducible components of  $\text{Per}(n, k)$ . It could be possible that the multiplicity of  $\text{Per}(n - k)$  as a component of  $\text{Per}(n, k)$  is so large that most of the mass of  $[\text{Per}(n, k)]$  would actually be concentrated on  $\text{Per}(n - k)$ .

Let  $\text{Preper}(n, k) \subset \text{Per}(n, k)$  be the closure of the subset of parameters where  $c(\lambda)$  is strictly preperiodic. More precisely,  $\text{Preper}(n, k)$  is locally defined in the open Zariski subset  $\bigcup_0^n \{f_\lambda^j c(\lambda) \neq c(\lambda)\}$  by  $f_\lambda^n c(\lambda) = f_\lambda^k c(\lambda)$ . It extends uniquely to an analytic subset of  $\Lambda$ . It is a hypersurface, which consists of a union of irreducible components of  $\text{Per}(n, k)$ , and endowed with the same multiplicities as  $\text{Per}(n, k)$ .

Because we allow arbitrary sequences  $0 \leq k(n) \leq n$ , we can strengthen the previous result as follows.

**Corollary 4.3.** *Assume that  $(\Lambda, f, c)$  is a family of rational maps satisfying (H1) or (H2) as in the preceding theorem. Let  $e \in \{0, 1\}$  be the generic cardinality of  $\mathcal{E}(f_\lambda)$ . Then for every fixed  $k \geq 1$ , we have*

$$\frac{1}{d^n + d^{(1-e)(n-k)}} [\text{Preper}(n, n - k)] \rightarrow T .$$

*Proof.* We assume  $e = 0$ , the other case is similar. If  $\lambda \in \text{Per}(n, n - k)$  and  $c(\lambda)$  is periodic, then its period divides  $k$ . So we may write

$$[\text{Per}(n, n - k)] = [D_n] + [\text{Preper}(n, n - k)],$$

where  $D_n$  is a divisor supported on  $\text{Per}(k)$ . We only need to prove that  $[D_n]/(d^n + d^{n-k}) \rightarrow 0$ . But if not,  $T$  would give some mass to the subvariety  $\text{Per}(k)$ . This is impossible as  $T$  has continuous potential.  $\square$

*Proof of Theorem 4.2 in the case where  $e = 0$ .* Let  $F_n : \Lambda \rightarrow \mathbb{P}^1$  be the map defined by  $F_n(\lambda) = f_\lambda^n(c(\lambda))$ . Recall from Proposition 3.1 that the bifurcation current is defined as the limit of the sequence  $F_n^* \omega$ , where  $\omega$  is the Fubini-Study (1,1) form on  $\mathbb{P}^1$ . We first prove

that the potentials of the currents  $[\text{Per}(n, k(n))] - (F_n^* \omega + F_{k(n)}^* \omega)$  converge to zero on the activity locus. A classical argument based on potential theory and which goes back to Brolin [B] allows us to derive the convergence of the sequence of currents everywhere on  $\Lambda$ .

We first note that  $(d^n + d^{k(n)})^{-1} ([\text{Per}(n, k(n))] - (F_n^* \omega + F_{k(n)}^* \omega)) = dd^c h_n$ , with

$$(6) \quad h_n := \frac{1}{d^n + d^{k(n)}} \log d_{\mathbb{P}^1} \left( f_\lambda^n c(\lambda), f_\lambda^{k(n)} c(\lambda) \right).$$

Here  $d_{\mathbb{P}^1}$  denotes the spherical distance on  $\mathbb{P}^1$ , given in homogeneous coordinates by

$$d_{\mathbb{P}^1}^2([z_0 : z_1], [w_0 : w_1]) = \frac{|z_0 w_1 - z_1 w_0|^2}{(|z_0|^2 + |z_1|^2)(|w_0|^2 + |w_1|^2)}.$$

Our aim is to show that  $\lim_n h_n = 0$ . Observe that  $d_{\mathbb{P}^1} \leq 1$ , whence  $\limsup_n h_n \leq 0$ .

The function  $h_n$  is not plurisubharmonic, but standard compactness results for families of psh functions still apply in this context. More precisely, Proposition 3.1 implies that  $(d^n + d^{k(n)})^{-1} (F_n^* \omega + F_{k(n)}^* \omega) = \omega + dd^c \tilde{g}_n$ , where  $\tilde{g}_n$  is a sequence of continuous functions converging uniformly on compact subsets to a continuous function  $g_\infty$ . Locally near any parameter we may write  $\omega$  as the  $dd^c$  of a smooth function, so that  $(d^n + d^{k(n)})^{-1} (F_n^* \omega + F_{k(n)}^* \omega) = dd^c v_n$ , where  $v_n$  is a sequence of continuous psh functions converging uniformly to a continuous psh function  $v_\infty$ .

Let  $u_n := h_n + v_n$ . It is an upper-semicontinuous function, and  $dd^c u_n$  is a positive closed current, hence  $u_n$  is a psh function. The sequence  $(v_n)$  converges uniformly, and  $\limsup_n h_n \leq 0$  so  $(u_n)$  is a sequence of psh functions which is locally uniformly bounded from above. In particular we have the following dichotomy, see [Ho, p.94]. Either  $u_n$  (hence  $h_n$ ) converges uniformly on compact subsets to  $-\infty$ ; or there exists a convergent subsequence  $u_{n_j} \rightarrow u$ , and  $u$  is again a psh function. In the latter case, we infer that  $h_{n_j} \rightarrow h := u - v_\infty$  and  $dd^c h = \lim_j dd^c h_{n_j}$ .

In any case, we may pick a subsequence, which we still denote by  $h_n$  converging to a function  $h$  which is either the sum of a continuous function and a psh function, or identically  $-\infty$ . The proof will be complete if we show that  $h = 0$ .

*Claim 1:* If  $f_{\lambda_0}^n c(\lambda_0)$  converges towards an attracting (or super-attracting) cycle, then  $h(\lambda_0) = 0$ . In particular  $h \not\equiv -\infty$  in this case.

Suppose by contradiction that  $h(\lambda_0) < 0$ . The Hartogs' Lemma [Ho, p.94] applied to the sequence of psh functions  $u_n$  and the function  $v_\infty$ , implies the existence of  $\varepsilon_0 > 0$  and a neighborhood  $U$  of  $\lambda_0$ , such that  $h_n|_U < -\varepsilon_0 < 0$  for infinitely many  $n$ . In particular,

$$(7) \quad d_{\mathbb{P}^1} \left( f_\lambda^n c(\lambda), f_\lambda^{k(n)} c(\lambda) \right) \leq \exp(-\varepsilon_0 d^n) \text{ for all } \lambda \in U.$$

Let  $V \subset U$  be a small open set such that  $c(\lambda)$  converges to an attracting (or super-attracting) cycle for  $\lambda \in V$ . As  $d_{\mathbb{P}^1}(f_\lambda^n c(\lambda), f_\lambda^{k(n)} c(\lambda)) \leq \exp(-\varepsilon_0 d^n)$ , the cycle is in fact superattracting with multiplicity  $d$ , hence totally invariant. For any  $\lambda \in V$ , we thus have  $\text{card } \mathcal{E}(f_\lambda) \geq 1$ , a contradiction.

*Claim 2:*  $h \equiv 0$  on the activity locus. In particular,  $h \not\equiv -\infty$  when the activity locus is non-empty.

Indeed by Proposition 2.4 and Claim 1, if  $\lambda_0$  belongs to the activity locus, there exists a sequence of parameters  $\lambda_n \rightarrow \lambda_0$  with  $h(\lambda_n) = 0$ . Since  $h$  is upper semicontinuous and non positive, we conclude that  $h(\lambda_0) = 0$ .

*Claim 3:*  $(d^n + d^{k(n)})^{-1}[\text{Per}(n, k(n))] \rightarrow T \equiv 0$  on the passivity locus.

We use the classification of Theorem 4. Let  $U$  be a connected open set in the passivity locus. If  $\text{Per}(n, k) \cap U = \emptyset$  for all  $n \neq k$  then there is nothing to prove. Otherwise,  $\text{Per}(n_0 + k_0, k_0) \cap U \neq \emptyset$  for some  $n_0, k_0 \geq 1$ . By assumption,  $c(\lambda)$  is not stably preperiodic, so case (2) of this proposition is excluded. If now for some  $\lambda \in U$ , the point  $c(\lambda)$  is attracted by a (super-)attracting cycle, then this behaviour persists throughout  $U$  and the first claim implies that  $h = 0$ , hence  $T = 0$ , in  $U$ . The remaining case is handled by the following lemma.

**Lemma 4.4.** *Let  $U$  be a connected open set where  $c(\lambda)$  stays in the domain of linearization of an irrational neutral cycle. Then  $\bigcup_{n>k} \text{Per}(n, k)$  is a (closed) subvariety in  $U$ . Furthermore for any compact subset  $K \subset U$ , there exists a constant  $C = C(K) < +\infty$  such that*

$$\sup_{n>k} \text{Mass}[\text{Per}(n, k)](K) \leq C .$$

Assuming the lemma for the moment, we proceed with the proof of the theorem.

*Claim 4:* the restriction of  $h$  to any germ of analytic curve is continuous (if not identically  $-\infty$ ).

The statement is local so let  $v$  be a local potential of  $T$  near a parameter  $\lambda$ . Because  $v$  is continuous,  $h$  is continuous iff  $\tilde{h} := h + v$  is. As  $dd^c(h + v)$  is a positive current,  $\tilde{h}$  is psh. On the support of  $T$ ,  $h$  is identically zero by the second claim, so  $\tilde{h}|_{\text{Supp}(T)}$  is continuous. The third claim shows that  $h$  is pluriharmonic on the complement of  $\text{Supp}(T)$ . We infer that  $\tilde{h}$  is a psh function which is continuous on the support of  $dd^c\tilde{h}$ . This is known to imply continuity only in dimension 1. So let  $\Gamma$  be any germ of analytic curve parameterized by a function  $\phi : \Delta \rightarrow \Lambda$ . The function  $\tilde{h} \circ \phi$  is subharmonic, and continuous on the support of its Laplacian. The continuity principle (see [Ts, p.54]) then implies that  $\tilde{h} \circ \phi$  is continuous.

We are now in position to conclude the proof of the theorem. When the activity locus is empty, the theorem follows from Claim (3). Otherwise, we pick a convergent subsequence  $h_n \rightarrow h$ . Claim (2) implies that  $h \neq -\infty$ . We need to show that  $h \equiv 0$ .

Pick  $\lambda_0 \in \Lambda$  and choose an analytic curve  $\Gamma$  containing  $\lambda_0$  which is either quasiprojective if (H1) is satisfied, or given by the condition (H2). Let  $\Omega$  be a connected component of the intersection of  $\Gamma$  with the passivity locus. The second, third and fourth claims imply that  $h$  is harmonic on  $\Omega$ , continuous on  $\overline{\Omega}$  and zero on the boundary.

Under the condition (H1), the maximum principle proved in [Ts, Theorem III.28] and applied to  $h$  gives immediately  $h \equiv 0$ . Under (H2), either the component  $\Omega$  is unbounded so by assumption it is contained in the set where  $f_\lambda^n c(\lambda)$  converges to an attracting periodic cycle, and  $h \equiv 0$ . Or  $\Omega$  is relatively compact in  $\Gamma$  and the minimum principle applied to  $h$  forces  $h$  to be identically zero on  $\Omega$ .

This shows that  $h|_\Gamma = 0$ , and so  $h = 0$  everywhere on  $\Lambda$ . This concludes the proof of the theorem in the case where  $e = 0$ .  $\square$

*Proof of Lemma 4.4.* The fact that  $V := \bigcup_{n>k} \text{Per}(n, k)$  is a closed subvariety was already proved in Theorem 4. To conclude the proof we need to show that for any point  $\lambda_0 \in V$ , the local multiplicity  $m(n, k, \lambda_0) \in \mathbb{N}$  at  $\lambda_0$  of the equation  $f_\lambda^n c(\lambda) = f_\lambda^k c(\lambda)$  is bounded uniformly on  $n$  and  $k$ . Our aim is to choose adequate coordinates both in the parameter space and the dynamical plane such that the computation of these multiplicities becomes simple. Note that this problem is local both in the parameter and dynamical space.

First, we may assume that for some  $n_0$ ,  $f_{\lambda_0}^{n_0} c(\lambda_0)$  is periodic for period  $k_0$ , otherwise  $m(n, k, \lambda_0) = 0$  for all  $n \neq k$ . Write  $g_\lambda = f_\lambda^{n_0}$ . Since  $c$  is passive, by choosing a suitable small neighborhood  $U \subset \Lambda$  containing  $\lambda_0$ , we have  $\text{Per}(n, k) \cap U \neq \emptyset$  only if  $n = n'n_0 + k_0 + r$  and  $k = k'n_0 + k_0 + r$  for some integers  $0 \leq r < k_0$  and  $n', k'$ . In particular, in terms of divisors we have

$$\left[ f_\lambda^n c(\lambda) = f_\lambda^k c(\lambda) \right] = \sum_{r=0}^{k_0-1} \left[ g_\lambda^{n'} (f_\lambda^r c(\lambda)) = g_\lambda^{k'} (f_\lambda^r c(\lambda)) \right],$$

taking multiplicities into account. We bound the multiplicities of the equations on the right hand side when  $r = 0$ , the arguments being analogous for  $r > 0$ .

Define  $p(\lambda) := f_\lambda^{k_0} c(\lambda)$ . Then  $g_{\lambda_0}$  has a fixed point at  $p(\lambda_0)$ . By reducing  $U$  if necessary, and changing coordinates in the parameter plane, we may suppose that  $p(\lambda_0) = 0$  and  $g_\lambda(0) = 0$  for all parameters. By assumption,  $g_\lambda$  is linearizable at 0 with a multiplier  $\mu = \exp(2i\pi\theta)$  with  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ , and  $p(\lambda)$  lies in the domain of linearization of  $g_\lambda$  for every  $\lambda \in U$ . In other words, we may assume that  $g_\lambda(z) = \mu z$  for all  $\lambda \in U$  and all  $z$  small enough.

Now choose coordinates  $\lambda = (\lambda_1, \dots, \lambda_d)$  around  $\lambda_0 = 0$ . Write the expansion into power series of  $p$  as  $p(\lambda) = P_l(\lambda) + O(|\lambda|^{l+1})$ , where  $P_l$  is a non-zero homogeneous polynomial of degree  $l$ . Then

$$g_\lambda^{n'} p(\lambda) - g_\lambda^{k'} p(\lambda) = (\mu^{n'} - \mu^{k'}) P_l(\lambda) + O(|\lambda|^{l+1}).$$

Since  $\mu$  is not a root of unity, we observe that the multiplicity of this expression is constant equal to  $l$ , independently of  $n'$  and  $k'$ . This concludes the proof of the lemma.  $\square$

*Proof of Theorem 4.2 in the case where  $e = 1$ .* The problem is purely local. By the discussion of Section 2.3, we may thus assume that  $f_\lambda$  is a polynomial for all  $\lambda$ . In this case, one easily checks that the current  $\widehat{T}$  can be defined in  $\Lambda \times \mathbb{C}$  by the formula

$$\widehat{T} = \lim_n dd_{\Lambda \times \mathbb{C}}^c \frac{1}{d^n} \log^+ |f_\lambda(z)|,$$

and that  $T = \lim_{n \rightarrow \infty} d^{-n} \log^+ |f_\lambda^n c(\lambda)|$  in this case. We then write

$$[\text{Per}(n, k)] = dd^c \log |f_\lambda^n c(\lambda) - f_\lambda^k c(\lambda)|,$$

and the convergence theorem thus amounts to proving  $\lim h_n = 0$  with

$$(8) \quad h_n := \frac{1}{d^n} \log \frac{|f_\lambda^n c(\lambda) - f_\lambda^k c(\lambda)|}{\max\{|f_\lambda^n c(\lambda)|, 1\}}.$$

For each  $\lambda$ , let  $K_\lambda$  be the filled-in Julia set of  $f_\lambda$ , i.e. the set of points of bounded orbit in  $\mathbb{C}$ . It is a compact subset of  $\mathbb{C}$ . Moreover, reducing  $\Lambda$  if necessary, it is included in a fixed disk of radius  $C > 0$ .

Suppose  $c(\lambda) \notin K_\lambda$ , and fix  $\varepsilon > 0$ . Then for  $n$  sufficiently large, and any  $k < n$ , we have  $|f_\lambda^k c(\lambda)| \leq \varepsilon |f_\lambda^n c(\lambda)|$ . So  $\lim h_n(\lambda) = 0$ . Otherwise  $c(\lambda) \in K_\lambda$ , and it is clear in this case that  $\limsup h_n(\lambda) \leq 0$ .

The proof now proceeds exactly like in the former case, through the proof of Claims 1, 2, 3, and 4. To adapt the proof of Claim 1, remark that since  $e < 2$ , any superattractive point at finite distance has multiplicity not greater than  $d - 1$ .  $\square$

### 5. PARAMETERIZING THE SPACE OF CRITICALLY MARKED POLYNOMIALS

Let  $\mathcal{P}_d$  be the space of all polynomials of degree  $d \geq 2$  with  $d-1$  marked critical points up to conjugacy by an affine transformation. A point in  $\mathcal{P}_d$  is represented by a  $d$ -tuple  $(P, c_0, \dots, c_{d-2})$  where  $P$  is a polynomial of degree  $d$ , and the  $c_i$ 's are complex numbers such that  $\{c_0, \dots, c_{d-2}\}$  is the set of all critical points of  $P$ . For each  $i$ ,  $\text{Card}\{j, c_j = c_i\}$  is the order of vanishing of  $P'$  at  $c_i$ . Two points  $(P, c_0, \dots, c_{d-2})$  and  $(P', c'_0, \dots, c'_{d-2})$  are identified when there exists an affine map  $\phi$  such that  $P' = \phi P \phi^{-1}$ , and  $c'_i = \phi(c_i)$  for all  $0 \leq i \leq d-2$ .

The set  $\mathcal{P}_d$  is a quasiprojective variety of dimension  $d-1$ , and is isomorphic to the quotient of  $\mathbb{C}^{d-1}$  by the finite group of  $(d-1)$ -th roots of unity acting linearly and diagonally on  $\mathbb{C}^{d-1}$  (see [Si] or Proposition 5.1 below). For instance,  $\mathcal{P}_3$  is isomorphic to the quadric cone  $\{z^2 = xy\} \subset \mathbb{C}^3$ . When  $d \geq 3$ , this space admits a unique singularity at the point  $(z^d, 0, \dots, 0)$ . Depending on the problem we consider, we shall work directly on  $\mathcal{P}_d$ , viewed as an abstract variety, or with an “orbifold parameterization” of this space by  $\mathbb{C}^{d-1}$ , which we describe shortly. Working with the parameterization is better suited for computing the masses of the bifurcation currents. On the other hand the description of the bifurcation measure in terms of external rays is simpler in  $\mathcal{P}_d$  (see Section 7).

We now describe our orbifold parameterization of  $\mathcal{P}_d$ .

Let first  $\tilde{\pi}$  be the map  $\tilde{\pi} : \mathbb{C}^{d-1} \rightarrow \mathcal{P}_d$  which maps  $(c_1, \dots, c_{d-2}, \alpha) \in \mathbb{C}^{d-1}$  to the primitive of  $z \prod_1^{d-2} (z - c_i)$  whose value at 0 is  $\alpha$ . For the sake of simplicity, we write  $c$  for  $(c_1, \dots, c_{d-2})$ , and for  $\alpha \in \mathbb{C}$ , we have

$$(9) \quad \tilde{P}_{c,\alpha}(z) = \frac{1}{d} z^d + \sum_{j=2}^{d-1} (-1)^{d-j} \sigma_{d-j}(c) \frac{z^j}{j} + \alpha ,$$

where  $\sigma_i(c)$  is the symmetric polynomial in  $\{c_j\}_1^{d-2}$  of degree  $i$ . The critical points of  $\tilde{P}_{c,\alpha}$  are  $\{0, c_1, \dots, c_{d-2}\}$ , so there is a natural map  $\tilde{\pi} : \mathbb{C}^{d-1} \rightarrow \mathcal{P}_d$ , which is a finite ramified cover.

For our purpose, it is better to work with a slightly different parameterization. For  $(c, a) \in \mathbb{C}^{d-2} \times \mathbb{C}$ , we define

$$P_{c,a} := \tilde{P}_{c,a^d} .$$

The critical set of  $P_{c,a}$  is again given by  $(0, c_1, \dots, c_{d-2})$ , so that we get a natural map  $\pi : \mathbb{C}^{d-1} \rightarrow \mathcal{P}_d$ ,  $(c, a) \mapsto (P_{c,a}, 0, c_1, \dots, c_{d-2})$ . The advantage of this parameterization is that all currents of bifurcation have the same mass, see Proposition 6.7 below. Both orbifold parameterizations of  $\mathcal{P}_d$  are very much inspired by the one described in [BrH1]. Notice however that they chose to have *centered* polynomials, while we choose to have a critical point at 0.

**Proposition 5.1.** *The natural map  $\pi : \mathbb{C}^{d-1} \rightarrow \mathcal{P}_d$  is a finite ramified cover of degree  $d(d-1)$ .*

*Its critical set is precisely  $\{(c, a), P_{c,a}(0) = 0\} = \{a = 0\}$ . The set of critical values of  $\pi$  is the set of polynomials with marked critical points  $(P, c_1, \dots, c_{d-1})$  such that  $P(c_1) = c_1$ .*

*Finally, the number of preimages of a critical value under  $\pi$  is 1 in the case of  $P(z) = z^d$  and  $d-1$  otherwise.*

*Proof.* We need to understand when two parameters  $(c, a)$  and  $(c', a')$  map to the same point in  $\mathcal{P}_d$ , i.e. when there exists an affine map  $\phi$  such that  $\phi \circ P_{c,a} \circ \phi^{-1} = P_{c',a'}$ ,  $\phi(c_i) = c'_i$  for all  $i$ , and  $\phi(0) = 0$ . The latter fact implies that  $\phi = \zeta z$  for some  $\zeta$ . But the polynomials  $P_{c,a}$ ,  $P_{c',a'}$  have the same leading monomial, so  $\zeta^{d-1} = 1$ . We conclude that  $\pi(c, a) = \pi(c', a')$  iff there exists a  $(d-1)$ -th root of unity  $\zeta$  such that  $\zeta c_i = c'_i$  for all  $i$ , and  $\zeta a^d = (a')^d$ . The proposition follows easily from this fact.  $\square$

## 6. HIGHER BIFURCATION CURRENTS AND THE BIFURCATION MEASURE

In this section, we work exclusively with the parameterization  $(c, a) \mapsto P_{c,a}$  of the space of polynomials with all critical points marked, as described before, and write  $c_0 := 0$ . We introduce higher bifurcation currents and the bifurcation measure. This leads in particular to the proof of Theorem 5.

**6.1. Basics.** Recall that the filled-in Julia set  $K(P)$  of a polynomial  $P$  is the set of points with bounded orbits. The filled-in Julia set is compact, and its boundary is the Julia set of  $P$ .

**Definition 6.1.** *The connectedness locus is the set of parameters in  $\mathbb{C}^{d-1}$  for which the filled-in Julia set of  $P_{c,a}$  is connected. Equivalently, it is the set of parameters for which all critical points have bounded orbit. We denote it by  $\mathcal{C}$ .*

The following fundamental result comes from [BrH1]:

**Proposition 6.2.** *The connectedness locus is compact in  $\mathbb{C}^{d-1}$ .*

Recall that the Green function of a polynomial  $P$  of degree  $d$  is by definition  $g_P = \lim_n d^{-n} \log^+ |P^n|$ . Let  $z \mapsto g_{P_{c,a}}(z)$  be the Green function of  $P_{c,a}$ . We define a function on the parameter space  $\mathbb{C}^{d-1}$  by putting  $G(c, a) = \max \{g_{P_{c,a}}(c_k), 0 \leq k \leq d-2\}$ . We will give more properties of  $G$  later on. The previous proposition is an obvious consequence of the following estimate, which is a result of [BrH1]. Because we work in different coordinates, we provide a detailed proof.

**Proposition 6.3.**  $G(c, a) = \log^+ \max\{|a|, |c_k|\} + O(1)$ .

The proof is based on the following two lemmas.

**Lemma 6.4.** *For all  $z \in \mathbb{C}$ , we have*

$$g_{P_{c,a}}(z) \leq \log \max\{|z|, A\} + \frac{\log C}{d-1},$$

for some constant  $C$  depending only on  $P$  and with  $A = \max\{|c_k|, |a|\}$ .

**Lemma 6.5.** *For all  $z \in \mathbb{C}$ , we have*

$$\max\{g_{P_{c,a}}(z), G(c, a)\} \geq \log |z - \delta| - \log 4,$$

where  $\delta = \sum c_k/(d-1)$ .

*Proof of Proposition 6.3.* Let  $A = \max\{|a|, |c_k|\}$ . Since  $|c_k| \leq A$  for all  $k$ , Lemma 6.4 yields  $g_P(c_k) \leq \log A + \log C/(d-1)$ . The same estimate holds for  $g_P(0)$ , whence

$$G(c, a) = \max\{g_{P_{c,a}}(0), g_{P_{c,a}}(c_k)\} \leq \log \max\{|c_k|, |a|\} + \log C/(d-1).$$

To get the estimate from below, we apply Lemma 6.5. Let  $A \geq 2$ . Assume first that all complex numbers  $\delta, c_1 - \delta, \dots, c_{d-2} - \delta$  have modulus  $< A/2$ , in which case  $A = a$ . Then by lemma 6.5 we get

$$d \times G(c, a) \geq \max \left\{ g_{P_{c,a}}(a^d), G(c, a) \right\} \geq \log |a^d - \delta| - \log 4 \geq d \log |A| - \log 8.$$

Here the first inequality follows from the fact that  $a^d = P_{c,a}(0)$  and the last one holds because  $|A^d - \delta| \geq A^d/2$ .

In the opposite case, among the complex numbers  $\delta, c_1 - \delta, \dots, c_{d-2} - \delta$ , at least one has modulus  $\geq A/2$ . So we deduce from lemma 6.5 that  $G(c, a) \geq \log \max\{|c_k|, |a|\} - \log 8$ .  $\square$

*Proof of Lemma 6.4.* By definition,  $P_{c,a} = \tilde{P}_{c,a^d}$ , and (9) yields

$$|P(z)| \leq d^{-1}|z|^d \left( 1 + d \times \max \left\{ \frac{\sigma_{d-j}(c)}{|z|^{d-j}}, \frac{|a|^d}{|z|^d} \right\} \right),$$

where  $\sigma_{d-j}(c)$  is the symmetric polynomial of degree  $d-j$  in the  $c_k$ 's. When  $|z| \geq A = \max\{|c_k|, |a|\}$ , we infer that  $|P(z)| \leq C|z|^d$  for a real number  $C$  depending only on  $d$ . By the maximum principle,  $|P(z)| \leq CA^d$  when  $|z| \leq A$ . These estimates classically imply the statement of the lemma.  $\square$

*Proof of Lemma 6.5.* Fix a polynomial  $P$ . Recall that the Böttcher coordinate is a *univalent* function  $\varphi_P : \{g_P > G(P)\} \rightarrow \mathbb{C}$  satisfying the functional equation  $\varphi_P \circ P(z) = \phi_P(z)^d$ . In particular,  $\log |\varphi_P| = g_P$  where the left hand side is defined. It is actually possible to choose a branch of the  $(d^n)$ <sup>th</sup>-root such that  $\varphi_P(z) := \lim_{n \rightarrow \infty} (P^n(z))^{1/d^n}$  where the convergence is uniform, see [Mi2]. When  $P = P_{c,a}$ , a direct induction shows that  $P^n(z) = z^{d^n} - \frac{d^n}{d-1} \sigma_1(c) z^{d^n-1} + \text{l.o.t.}$ , so that  $\varphi(z) = z - \delta + O(1/z)$  with  $\varphi := \varphi_{P_{c,a}}$  and  $\delta := \sigma_1(c)/(d-1) = \sum c_k/(d-1)$ . Let  $\psi := \varphi^{-1}$ . It is a univalent function on  $\mathbb{C} \setminus \overline{D}(0, \exp G(c, a))$  and  $\psi(z) = z + \delta + O(1/z)$  at infinity.

We can thus apply [BrH1, Corollary 3.3] –which is a version of the Koebe 1/4-Theorem– to the univalent function  $\psi - \delta$ . This yields  $\psi(\mathbb{C} \setminus \overline{D}(0, r)) \supset \mathbb{C} \setminus \overline{D}(\delta, 2r)$  when  $r > \exp G(c, a)$ . Now pick any  $z \in \mathbb{C}$ , and write  $g_P(z) = \log r$ . Then, since  $|\phi(z)| = r$ ,  $z \notin \psi(\mathbb{C} \setminus \overline{D}(0, 2 \max\{r, \exp G(c, a)\}))$ , thus  $|z - \delta| \leq 4 \max\{r, \exp G(c, a)\}$ . We conclude by taking logarithms in both sides.  $\square$

We close this paragraph by introducing some terminology.

**Definition 6.6.** A polynomial is said to be *critically finite* if all its critical points are preperiodic. It is of *Misiurewicz type* if all its critical points are mapped to repelling periodic orbits.

It is a classical fact that if all critical points are strictly preperiodic, then the polynomial is Misiurewicz (see [CG, p.92]).

**6.2. Currents and measure of bifurcation.** We now define  $d-1$  positive closed  $(1,1)$  currents on the parameter space  $\mathbb{C}^{d-1}$ , each describing the bifurcation of one critical point. This part is essentially a rephrasing of the results of Section 3 in this specific situation.

As before, consider the Green function  $z \mapsto g_{P_{c,a}}(z)$  of the polynomial  $P_{c,a}$ . It is a continuous positive subharmonic function, and the filled-in Julia set of  $P_{c,a}$  equals  $\{g_{P_{c,a}} = 0\}$ . The function  $((c, a), z) \mapsto g_{P_{c,a}}(z)$  is psh in  $\mathbb{C}^{d-1} \times \mathbb{C}$ . In particular, for all  $0 \leq k \leq d-2$ ,  $g_k := g_{P_{c,a}}(c_k)$  induces a psh function on  $\mathbb{C}^{d-1}$ .

Let  $\omega$  be the Fubini Study metric on  $\mathbb{C}^{d-1}$ . If  $T$  is a positive closed current of bidegree  $(k, k)$  on  $\mathbb{C}^{d-1}$ , we let  $\text{Mass}(T)$  be its *mass* relative to  $\omega$ , that is

$$\text{Mass}(T) = \int_{\mathbb{C}^{d-1}} T \wedge \omega^{d-1-k}.$$

It is convenient to write  $d^c = (\partial - \bar{\partial})/2i\pi$ , so that if  $z = (z_1, \dots, z_{d-1}) \in \mathbb{C}^{d-1}$ , the mass of  $dd^c \log |z_1|$  is 1. In terms of potentials, if  $u = \log^+ \max\{|z_i|\} + O(1)$ , then the mass of  $dd^c u$  is again 1. We also repeatedly use the following Bézout-type estimate (see for instance [FS1]):

if  $T_1, \dots, T_q$  are positive closed currents with finite mass and locally bounded potentials (so that their wedge product is well defined, see [BT1, De]), then

$$\text{Mass}(T_1 \wedge \dots \wedge T_q) \leq \text{Mass}(T_1) \cdots \text{Mass}(T_q).$$

For  $0 \leq k \leq d-2$ , we set  $T_k = dd^c g_k = dd^c g_{P_{c,a}}(c_k)$ .

**Proposition 6.7.** *For each  $0 \leq k \leq d-2$ ,  $T_k$  is a positive closed current of bidegree  $(1,1)$ , with continuous potential and mass 1 in  $\mathbb{C}^{d-1}$ . The support of  $T_k$  is precisely the activity locus of the critical point  $c_k$ , and is equal to the boundary of the closed set  $\{g_{P_{c,a}}(c_k) = 0\}$ .*

By summing these currents, we get a current whose support is the bifurcation locus. This is the *bifurcation current*, originally considered in [DeM1]. It was proved to equal  $dd^c \text{Lyap}(P)$  in [DeM2] for families of rational maps. For convenience, we restate and reprove her results in our context of polynomial maps.

**Proposition 6.8** ([DeM1, DeM2]). *Define  $T_{\text{bif}} := (d-1)^{-1} \sum_0^{d-2} T_k$ . This is a positive closed  $(1,1)$  current of mass 1, whose support is precisely the complement of the set of polynomials structurally stable on their Julia sets. Further,  $T_{\text{bif}} = d^{-1} dd^c \text{Lyap}(P_{c,a})$ , where  $\text{Lyap}(P)$  is the Lyapunov exponent of  $P$  relative to its measure of maximal entropy.*

We can now define higher dimensional bifurcation currents by intersecting the currents  $T_k$ . The currents  $T_k$  admit locally continuous potentials, so their intersections are well-defined. We first note the following fact.

**Proposition 6.9.** *For each  $k$ , we have  $T_k \wedge T_k = 0$ .*

**Definition 6.10.** *For each  $1 \leq l \leq d-1$ , we define the  $l$ -bifurcation current to be:*

$$T_{\text{bif}}^l = \frac{(d-l-1)!}{(d-1)!} \left( \sum_{i=0}^{d-2} T_i \right)^{\wedge l}$$

*It is a positive closed current of bidegree  $(l,l)$ .*

These higher currents of bifurcation were first considered by [BB1] in the more general context of rational maps.

**Proposition 6.11.** *The current  $T_{\text{bif}}^l$  is a non zero positive closed  $(l,l)$  current, and its trace measure does not charge pluripolar (hence analytic) sets. It has finite mass 1, and its support is included in the set of polynomial for which (at least)  $l$  critical points are active.*

*Remark 6.12.* As we noticed in the introduction, for  $l > 1$  the support of  $T_{\text{bif}}^l$  is not equal to the set where  $l$  critical points are active. The following example was indicated to us by A. Douady, and was studied in great detail in the Ph. D. thesis of P. Willumsen [W].

*Example 6.13.* Take  $P = z + 1/2 z^2 + z^3$ . Any cubic polynomial  $Q$  close enough to  $P$  has one or two fixed points near 0. In case there are two fixed points, denote by  $\mu_1$  and  $\mu_2$  their multipliers. The residue of  $1/(P - z)$  at 0 equals  $-4$ , so  $(\mu_1 - 1)^{-1} + (\mu_2 - 1)^{-1}$  is close to  $-4$ . It is not difficult to check that in this case, either  $\mu_1$  or  $\mu_2$  has modulus  $< 1$  (see [Mi2]), so the associated fixed point attracts one critical point. Therefore, by Proposition 6.11 such a  $Q$  cannot lie in the support of the measure  $T_{\text{bif}}^2$ .

Hence near  $P$ , the measure  $T_{\text{bif}}^2$  has support in the subvariety of parameters with a fixed point of multiplier 1. Since this measure does not charge curves,  $P$  does not lie in the support either.

On the other hand, the two critical points of  $P$  are complex conjugate, so both of them are attracted by the parabolic fixed point 0. At least one of them is active: indeed if not, for nearby parameters we would have a critical point attracted by a repelling fixed point. By symmetry, we conclude that both are active.

For  $l = d - 1$ , we get a positive measure that we denote by  $\mu_{\text{bif}}$ . Its study will be our main focus in the sequel. The following proposition summarizes its first properties.

**Proposition 6.14.**

- $\mu_{\text{bif}} = T_0 \wedge \cdots \wedge T_{d-1}$ .
- *The measure  $\mu_{\text{bif}}$  is a positive probability measure, supported on the boundary of the connectedness locus  $\mathcal{C}$ .*
- *It does not charge pluripolar sets. In particular, it does not charge analytic subsets.*
- *It is the pluricomplex equilibrium measure of the compact set  $\mathcal{C} \subset \mathbb{C}^{d-1}$ . In particular, its support is the Shilov boundary of  $\mathcal{C}$ .*

We refer to [Kl] for basic notions in pluripotential theory, such as “pluripolar sets”, or “equilibrium measure”. See [BT2] for the notion of Shilov boundary and the fact that it equals  $\text{Supp}(\mu_{\text{Bif}})$ . The last statement of the proposition says in particular that  $\mu_{\text{Bif}}$  is natural from the point of view of complex analysis.

*Proof of Proposition 6.7.* By definition,  $T_k = dd^c g_k$  where  $g_k = g_{P_{c,a}}(c_k)$  is psh and continuous. The continuity of  $g_k$  is classical in this context, or follows directly from Proposition 3.1. The support of  $T_k$  is the activity locus of  $c_k$  by Theorem 3.2.

Assume now that  $g_k(c, a) > 0$ . Then  $c_k$  is attracted towards the superattracting point  $\infty$ , hence  $c_k$  is passive at  $(c, a)$ , so  $T_k = 0$  near  $(c, a)$ . It is also clear that  $T_k = 0$  in the interior of  $\{g_k = 0\}$ , so that  $\text{Supp } T_k \subset \partial\{g_k = 0\}$ . Conversely, by the maximum principle,  $g_k$  cannot be pluriharmonic in an open set  $U$  intersecting  $\partial\{g_k = 0\}$ . We conclude that  $\text{Supp } T_k = \partial\{g_k = 0\}$ .

It remains to compute the mass of the current  $T_k$ . Proposition 6.3 asserts that  $g_k(c, a) \leq \log \max\{|c_k|, |a|\} + O(1)$  so  $\text{Mass}(T_k) \leq 1$ .

If now  $L$  is a complex line in  $\mathbb{C}^{d-1}$  and  $T$  is a positive current of bidegree (1,1) with continuous potential, we define the restriction  $T|_L$  of  $T$  to  $L$  by first restricting the potential and then taking  $dd^c$ . The restriction  $T|_L$  is a positive measure on  $L$ , and  $\text{Mass}(T|_L) \leq \text{Mass}(T)$  by Bezout ( $T|_L = T \wedge [L]$ ).

To get the opposite inequality for  $\text{Mass}(T_k)$ , we restrict  $T_k$  to the line of unicritical polynomials  $L := \{c_1 = \dots = c_{d-2} = 0\}$ . It is enough to prove that  $\text{Mass}(T_k|_L) = 1$ . With our parameterization, unicritical polynomials are of the form  $\frac{1}{d}z^d + a^d$ . For every  $k$ ,  $g_k(0, a)$  equals  $g_a(0)$ , where  $g_a$  is the Green function of  $\frac{1}{d}z^d + a^d$ . From proposition 6.3 we infer that  $g_a(0) = \log|a| + O(1)$  at infinity, so the mass of the restriction is 1.  $\square$

*Proof of Proposition 6.8.* All currents  $T_k$  have mass 1, hence  $T_{\text{bif}}$  also has mass 1.

By [MSS], a polynomial is structurally stable on  $J$  iff all its critical points are passive. Using Proposition 6.7, we conclude that a polynomial is structurally unstable iff it belongs to the union of the support of the  $T_k$ 's. The latter is precisely the support of  $T_{\text{bif}}$ .

The Lyapunov exponent of a polynomial  $P$  can be computed in terms of the Green function, see [Ma, P1]. We get  $\text{Lyap}(P_{c,a}) = \log d + \sum_0^{d-2} g_{P_{c,a}}(c_k)$ . We already know that  $dd^c g_{P_{c,a}}(c_k) = T_k$ , whence  $dd^c \text{Lyap}(P_{c,a}) = \sum_0^{d-2} T_k = (d-1) \times T_{\text{bif}}$ .  $\square$

*Proof of Proposition 6.9.* As before write  $g_k = g_{P_{c,a}}(c_k)$ , so that  $dd^c g_k = T_k$ . The sequence of continuous psh functions  $\max\{g_k, \varepsilon\}$  decreases to  $g_k$  when  $\varepsilon$  decreases to zero. From [BT1], we infer that  $dd^c \max\{g_k, \varepsilon\} \wedge dd^c g_k$  converges weakly to  $dd^c g_k \wedge dd^c g_k = T_k \wedge T_k$ . Since  $g_k$  is pluriharmonic where it is positive, the support of  $dd^c \max\{g_k, \varepsilon\} \wedge dd^c g_k$  is contained in the intersection  $\{g_k = \varepsilon\} \cap \{g_k = 0\}$  which is empty. Hence  $T_k \wedge T_k = 0$ .  $\square$

*Proof of Proposition 6.11.* The fact that  $T_{\text{bif}}^l$  is a positive closed  $(l, l)$  current whose trace measure does not charge pluripolar set is a consequence of the definition and the fact that all currents  $T_l$  have continuous potentials, see [De].

Take a polynomial  $P$  for which at most  $l - 1$  critical points are active. Then  $T_j = 0$  for at least  $d - l$  distinct integers  $j$ . Combining this with Proposition 6.9, we conclude that  $T_{\text{bif}}^l = 0$  near  $P$ . The support of the  $l$ -th bifurcation current is thus contained in the set of polynomials with at least  $l$  active critical points.

It remains to compute the mass of  $T_{\text{bif}}^l$  for every  $2 \leq l \leq d - 1$ . To this end, we rely on the following computation.

**Lemma 6.15.** *Let  $I = (i_1, \dots, i_l)$  be a multi-index with  $l$  distinct entries in  $\{0, \dots, d - 2\}$ . Then*

$$(10) \quad T_{i_1} \wedge \cdots \wedge T_{i_l} = (dd^c)^l G_I \text{ with } G_I = \max\{g_{i_1}, \dots, g_{i_l}\} .$$

Let us continue with the proof of the proposition. By Lemma 6.15, we have that  $(dd^c)^{d-1} G = T_0 \wedge \cdots \wedge T_{d-2}$  where  $G = \max\{g_0, \dots, g_{d-2}\}$ . Recall from Proposition 6.3 that  $G(c, a) = \log \max\{|c_k|, |a|\} + O(1)$ . Standard estimates in pluripotential theory (see e.g. [Kl, p. 212]) imply that the measure  $(dd^c)^{d-1} G$  has unit mass. Let now  $i_1, \dots, i_{d-1}$  be any reordering of the  $d - 1$  integers  $\{0, \dots, d - 2\}$ . Then

$$\begin{aligned} 1 = \text{Mass}(T_{i_1} \wedge \cdots \wedge T_{i_{d-1}}) &\leq \text{Mass } T_{i_{d-1}} \times \text{Mass}(T_{i_1} \wedge \cdots \wedge T_{i_{d-2}}) \\ &= \text{Mass}(T_{i_1} \wedge \cdots \wedge T_{i_{d-2}}) \leq \prod_{j=1}^{d-1} \text{Mass } T_{i_j} = 1 . \end{aligned}$$

Here the first inequality is the Bézout-type estimate for currents. We conclude that  $\text{Mass}(T_{i_1} \wedge \cdots \wedge T_{i_{d-2}}) = 1$ . Proceeding by descending induction, we infer that for any choice of distinct integers  $\{i_1, \dots, i_l\} \subset \{0, \dots, d - 2\}$ , the mass of  $(T_{i_1} \wedge \cdots \wedge T_{i_l})$  equals 1. Whence

$$\text{Mass} \left( \sum_0^{d-2} T_i \right)^{\wedge l} = \sum_{i_1 \neq i_2 \dots \neq i_l} \text{Mass}(T_{i_1} \wedge \cdots \wedge T_{i_l}) = (d-1)(d-2) \cdots (d-l) = \frac{(d-1)!}{(d-1-l)!} .$$

This concludes the proof of the proposition.  $\square$

*Proof of Lemma 6.15.* We prove that  $T_0 \wedge \cdots \wedge T_l = (dd^c)^{l+1} G_l$  with  $G_l = \max\{g_0, \dots, g_l\}$  for all  $l \leq d - 1$ . The same proof gives the general equality (10).

We proceed by induction on  $l$ . The statement is clear for  $l = 0$ . Suppose that we know that  $T_0 \wedge \cdots \wedge T_{l-1} = (dd^c)^l G_{l-1}$ . We want to compute  $T_0 \wedge \cdots \wedge T_l$ . We proceed as follows:

$$\begin{aligned} T_0 \wedge \cdots \wedge T_{l-1} \wedge T_l &= dd^c \left( g_l (dd^c)^l G_{l-1} \right) = dd^c \left( G_l (dd^c)^l G_{l-1} \right) \\ &= dd^c \left( G_{l-1} (dd^c)^{l-1} G_{l-1} \wedge dd^c G_l \right) \\ &= dd^c \left( G_l (dd^c)^{l-1} G_{l-1} \wedge dd^c G_l \right) = (dd^c)^{l+1} G_l . \end{aligned}$$

Let us justify these sequence of equalities. The first one is by definition. For the second one, observe that the support of  $(dd^c)^l G_{l-1}$  is contained in the intersection of the supports of  $T_0, \dots, T_{l-1}$ , and  $g_0 = \dots = g_{l-1} = 0$  on this set. So  $g_l = G_l$  there. The third one is just reordering the wedge product. For the fourth, note that on  $\{G_{l-1} < G_l\}$  we have  $G_l = g_l > 0$ . Thus  $G_l$  is pluriharmonic and  $dd^c G_l = 0$ . The last equality is obtained by repeating the same argument  $l - 1$  times.  $\square$

*Proof of Proposition 6.14.* The equation  $\mu_{\text{bif}} = T_0 \wedge \dots \wedge T_{d-1}$  is a consequence of Proposition 6.9. The first two facts are consequences of Proposition 6.11. It remains to prove the last one.

Consider the function  $G = \max g_k$ . This is a continuous psh function on  $\mathbb{C}^{d-1}$ , and  $\{G = 0\} = \mathcal{C}$ . We claim that

$$(11) \quad G = \sup \{u \text{ psh}, u - \log^+ \max\{|a|, |c_k|\} \leq O(1), u \leq 0 \text{ on } \mathcal{C}\} .$$

In the terminology of [Kl],  $G$  is the pluricomplex Green function of  $\mathcal{C}$  with pole at infinity. The equilibrium measure of  $\mathcal{C}$  is by definition the Monge-Ampère measure of  $G$ . It thus equals  $(dd^c)^{d-1}G = \mu_{\text{bif}}$  by (10).

The proof of (11) is standard, we include it for completeness. Take any psh function  $u$  with  $u \leq 0$  on  $\mathcal{C}$  and  $u - \log^+ \max\{|a|, |c_k|\} \leq O(1)$ . Choose  $\varepsilon > 0$ , and set  $u_\varepsilon := (1 - \varepsilon)u - \varepsilon$ . Then  $u_\varepsilon < G$  on a neighborhood of  $\mathcal{C}$  as  $G \leq 0$ ; this also holds in the complement of some ball  $B(0, R)$  by Proposition 6.3. Now pick any parameter  $(c, a) \notin \mathcal{C}$ . Consider an open set  $\Omega \subset \mathbb{C}^{d-1} \setminus \mathcal{C}$ ,  $(c, a) \in \Omega$ , with boundary contained in  $U \cup \mathbb{C}^{d-1} \setminus B(0, R)$ , where  $U$  is some small neighborhood of  $\mathcal{C}$ . On  $\Omega$  we have  $(dd^c)^{d-1}G = 0$ , so  $G$  is maximal as a psh function, see [Kl]. This implies that  $u_\varepsilon \leq G$  on  $\Omega$ . We have thus proved that  $u_\varepsilon \leq G$  everywhere. By letting  $\varepsilon \rightarrow 0$ , we conclude that  $u \leq G$ .  $\square$

**6.3. Density of Misiurewicz points.** We now aim at proving Theorem 5 cited in the introduction. It will be a consequence of the following more general result.

**Theorem 6.16.** *Choose a reordering  $\{i_0, \dots, i_{d-2}\}$  of  $\{0, \dots, d-2\}$  and fix  $0 \leq l \leq d-2$ . For each  $0 \leq j < l$ , let  $(n_j, m_j)$  be any couple of integers such that  $n_j > m_j$ . For  $l = 0$ , set  $W = \mathbb{C}^{d-1}$ , and for  $l \geq 1$ , let  $W \subset \mathbb{C}^{d-1}$  be the analytic subvariety (not necessarily reduced) consisting of parameters  $c, a$  for which  $P_{c,a}^{n_j}(c_{i_j}) = P_{c,a}^{m_j}(c_{i_j})$  for all  $0 \leq j < l$ . Then, the following hold.*

- (1) All irreducible components of  $W$  have codimension  $l$ .
- (2) Pick any  $n > k$ , and denote by  $\text{Per}(n, k)$  the set of polynomials such that  $P^n(c_{i_l}) = P^k(c_{i_l})$ . Then the variety  $\text{Per}(n, k) \cap W$  is a hypersurface in  $W$ .
- (3) For any sequence of integers  $k(n) < n$ , one has

$$\lim_{n \rightarrow \infty} \frac{1}{d^n} [\text{Per}(n, k(n)) \cap W] = T_{i_l} \wedge [W] .$$

in the weak topology of currents.

**Corollary 6.17.** *Fix a collection of distinct integers  $i_0, \dots, i_l \in \{0, \dots, d-2\}$ , and for any  $(n_0, \dots, n_l) \in (\mathbb{N}^*)^l$  choose a collection of integers  $k(n_0) < n_0, \dots, k(n_l) < n_l$ . Then define the analytic subset  $W_{n_0, \dots, n_l} = \bigcap_{j=1}^l [P^{n_j}(c_{i_j}) = P^{k(n_j)}(c_{i_j})]$ .*

*Then  $W_{n_0, \dots, n_l}$  has pure codimension  $l + 1$  and*

$$(12) \quad \lim_{n_l \rightarrow \infty} \cdots \lim_{n_1 \rightarrow \infty} \lim_{n_0 \rightarrow \infty} \frac{1}{d^{n_l + \dots + n_1 + n_0}} [W_{n_0, \dots, n_l}] = T_{i_l} \wedge \cdots \wedge T_{i_1} \wedge T_{i_0} .$$

With this result in hand, Theorem 5 follows easily:

*Proof of Theorem 5.* The current  $T_{\text{bif}}^l$  is a multiple of the positive closed  $(l, l)$  current  $\left(\sum_0^{d-2} T_i\right)^{\wedge l} = \sum_{i_1 \neq i_2 \dots \neq i_l} T_{i_1} \wedge \dots \wedge T_{i_l}$ . The theorem is then a direct consequence of Corollary 6.17.  $\square$

From this we also deduce Corollary 6. Notice that a critically finite polynomial is hyperbolic iff its critical points are all *periodic*. Recall that a polynomial is Misiurewicz if all critical points are preperiodic to repelling periodic orbits.

**Corollary 6.18.** *There exists a sequence of atomic measures supported on the set of Misiurewicz parameters (resp. on critically finite hyperbolic parameters) and converging to  $\mu_{\text{bif}}$ . In particular, the support of  $\mu_{\text{bif}}$  is contained in the closure of the set of Misiurewicz points (resp. of hyperbolic critically finite parameters).*

*Proof.* Corollary 6.17 implies that

$$\lim_{n_{d-2} \rightarrow \infty} \dots \lim_{n_0 \rightarrow \infty} \frac{1}{d^{n_{d-2} + \dots + n_0}} \left[ \bigcap_{j=0}^{d-2} \{P_{c,a}^{n_j}(c_j) = c_j\} \right] = T_0 \wedge \dots \wedge T_{d-2} .$$

On the left hand side we have a sequence of atomic measures supported on critically finite and hyperbolic parameters. The right hand side is  $\mu_{\text{bif}}$  by Proposition 6.14.

The same Corollary 6.17 again implies that

$$\lim_{n_{d-2} \rightarrow \infty} \dots \lim_{n_0 \rightarrow \infty} \frac{1}{d^{n_{d-2} + \dots + n_0}} \left[ \bigcap_{j=0}^{d-2} \{P_{c,a}^{n_j}(c_j) = P_{c,a}^{n_j-1}(c_j)\} \right] = T_0 \wedge \dots \wedge T_{d-2} .$$

To conclude the proof we use the same argument as in Corollary 4.3. Each divisor  $H_j := \{P_{c,a}^{n_j}(c_j) = P_{c,a}^{n_j-1}(c_j)\}$  can be decomposed as a sum of two divisors  $H_j = \text{Preper}_j + \text{Fix}_j$  where  $\text{Preper}_j$  is the set of points in  $H_j$  for which  $c_j$  is *strictly* preperiodic, and  $\text{Fix}_j$  is supported on the set on which  $c_j$  is *fixed*. As  $T_j$  does not charge hypersurfaces, we have  $d^{-n_j} \text{Preper}_j \rightarrow T_j$ , therefore

$$\lim_{n_{d-2} \rightarrow \infty} \dots \lim_{n_0 \rightarrow \infty} \frac{1}{d^{n_{d-2} + \dots + n_0}} \left[ \bigcap_{j=0}^{d-2} \text{Preper}_j \right] = T_0 \wedge \dots \wedge T_{d-2} .$$

The supports of the atomic measures on the left hand side are contained in the set of Misiurewicz parameters.  $\square$

*Proof of Corollary 6.17.* The proof is by induction on  $l$ . For  $l = 0$ , this is Theorem 4.2 for the critical point  $c_{i_0}$  (or equivalently, the statement (3) of Theorem 6.16 with  $l = 0$ ). Suppose now that it is true for some integer  $l$ , and pick  $i_{l+1} \in \{0, \dots, d-2\}$ . For  $k(n_{l+1}) < n_{l+1}$ , define  $\text{Per}(n_{l+1}, k(n_{l+1}))$  as the set of  $(c, a) \in \mathbb{C}^{d-1}$  such that  $P_{c,a}^{n_{l+1}}(c_{l+1}) = P_{c,a}^{k(n_{l+1})}(c_{l+1})$ . For ease of notation we write  $n$  for  $(n_0, \dots, n_l)$ . By Theorem 6.16,  $W_{n,n_{l+1}} = \text{Per}(n_{l+1}, k(n_{l+1})) \cap W_n$  has pure codimension  $l+2$ , and  $\lim_{n_{l+1} \rightarrow \infty} d^{-n_{l+1}} [W_{n,n_{l+1}}] = T_{i_{l+1}} \wedge [W_n]$ . Now, our inductive hypothesis asserts that  $d^{-(n_l + \dots + n_0)} [W_n] \rightarrow T_{i_l} \wedge \dots \wedge T_{i_1} \wedge T_{i_0}$ . Since,  $T_{i_{l+1}}$  has *continuous potential*, we deduce (see [De]) that

$$T_{i_{l+1}} \wedge d^{-(n_l + \dots + n_0)} [W_n] \rightarrow T_{i_{l+1}} \wedge T_{i_l} \wedge \dots \wedge T_{i_0}$$

as  $n_0, \dots, n_l \rightarrow \infty$ . By the convergence Theorem 4.2 we get that  $d^{-n_{l+1}} \text{Per}(n_{l+1}, k(n_{l+1})) \rightarrow T_{l+1}$  as  $n_{l+1} \rightarrow \infty$ , whence

$$d^{-n_{l+1}} \text{Per}(n_{l+1}, k(n_{l+1})) \wedge d^{-(n_l + \dots + n_0)} [W_n] \longrightarrow T_{i_{l+1}} \wedge T_{i_l} \wedge \dots \wedge T_{i_0},$$

as  $n_0, \dots, n_{l+1} \rightarrow \infty$ . The corollary is proved.  $\square$

*Proof of Theorem 6.16.* Pick any set of integers  $n_1 > m_1, \dots, n_{d-1} > m_{d-1}$ , and let  $W^k := \{(c, a), P_{c,a}^{n_j}(c_{i_j}) = P_{c,a}^{m_j}(c_{i_j}) \text{ for all } 1 \leq j \leq k\}$ . Being defined by  $k$  equations, the codimension of  $W^k$  is greater than or equal to  $k$ . For a polynomial lying in  $W^{d-1}$ , all critical points are preperiodic, whence  $P_{c,a}$  lies in the connected locus which is bounded by Proposition 6.2. So  $W^{d-1}$  is a finite set of points, i.e.  $\text{codim}_{\mathbb{C}} W^{d-1} = d - 1$ . By induction we conclude that  $\text{codim}_{\mathbb{C}} W^k = k$  for all  $k$ .

This proves (1) and (2). Theorem 4.2 implies (3), because the variety  $W$  in the statement of the theorem is quasi-projective.  $\square$

## 7. EXTERNAL RAYS

In this section we prove Theorems 8, 9 and 10. These rely on the combinatorial description of polynomials in terms of external rays landing at critical points. This technique was introduced and studied in [Go, BFH, K2]. We describe the set  $\mathbf{Cb}$  of all these combinatorics in detail in Section 7.1 and show that it is a compact and connected set endowed with a natural measure. We do not claim originality here, but we hope that our presentation will shed some light on the structure of this space. Following [Go] we construct in Section 7.2 a natural map  $\Phi_g$  from  $\mathbf{Cb} \times \mathbb{R}_+^*$  into  $\mathcal{P}_d$ . The space  $\mathbf{Cb} \times \mathbb{R}_+^*$  has a natural structure of Riemann surface lamination, and  $\Phi_g$  provides an embedding into  $\mathcal{P}_d \setminus \mathcal{C}$  preserving the lamination structure (Proposition 7.12). We then describe the extension of  $\Phi_g$  to a subset of  $\mathbf{Cb} \times \{0\}$  of full measure, and we state Kiwi's Continuity Theorem saying that  $\Phi_g$  extends *continuously* at Misiurewicz combinatorics. The restriction of  $\Phi_g$  to  $\mathbf{Cb} \times \{0\}$  defines a measurable "landing map" from  $\mathbf{Cb}$  into the boundary of the connectedness locus, which transports the natural measure on the combinatorial space onto the bifurcation measure (Section 7.3). In Section 7.4, we describe a connectedness property of a subset of the boundary of the connectedness locus containing  $\text{Supp}(\mu)$ . Section 7.5 is devoted to the proof of Theorem 9. Finally, in Section 7.6 we prove Theorem 10.

**7.1. The combinatorial space.** We describe the set  $\mathbf{Cb}$  of combinatorics of polynomials of degree  $d$  such that  $g_P$  takes the same value on all critical points. We define this space abstractly and study its geometry.

**7.1.1. The restricted combinatorial space.** We first look at the subset  $\mathbf{Cb}_0$  of combinatorics of polynomials of degree  $d$  for which all critical points are marked and simple. In order to define it formally, we first need the following

**Definition 7.1.** We denote by  $\mathbf{S}$  the set of pairs  $\{\alpha, \alpha'\}$  contained in the circle  $\mathbb{R}/\mathbb{Z}$ , such that  $d\alpha = d\alpha'$  and  $\alpha \neq \alpha'$ .

Two finite and disjoint subsets  $\theta_1, \theta_2 \subset \mathbb{R}/\mathbb{Z}$  are said to be *unlinked* if  $\theta_2$  is included in a single connected component of  $(\mathbb{R}/\mathbb{Z}) \setminus \theta_1$ .

**Definition 7.2.** We let  $\mathbf{Cb}_0$  be the set of  $(d-1)$ -tuples  $\Theta = (\theta_1, \dots, \theta_{d-1}) \in \mathbf{S}^{d-1}$  such that for all  $i \neq j$ , the two pairs  $\theta_i$  and  $\theta_j$  are disjoint and unlinked.

We spend the remaining part of this section describing the topology of  $S$  and  $Cb_0$ .

Recall that a smooth manifold of dimension  $n$  is a *translation manifold* if it admits an atlas for which the transition maps are translations in  $\mathbb{R}^n$ . Any translation manifold is endowed with a natural metric coming from the standard euclidean metric on  $\mathbb{R}^n$ , and is naturally oriented. It is thus endowed with a natural volume form, and a natural smooth measure.

Let us explain how to define a translation structure on  $Cb$ . The circle  $\mathbb{R}/\mathbb{Z}$  admits a natural structure of translation manifold, with a metric  $d_{\mathbb{R}/\mathbb{Z}}$ . Let  $d_h$  be the Hausdorff metric on compact sets in  $(\mathbb{R}/\mathbb{Z}, d_{\mathbb{R}/\mathbb{Z}})$ . It endows  $S$  with a structure of a compact metric space. Denote by  $\delta : S \rightarrow \mathbb{R}_+^*$  the map sending  $\theta = \{\alpha, \alpha'\}$  to the distance between  $\alpha$  and  $\alpha'$  in  $\mathbb{R}/\mathbb{Z}$ . This map is continuous with values in the discrete set  $\{1/d, \dots, [d/2]/d\}$  where  $[d/2]$  is the integral part of  $d/2$ . One easily checks that  $\delta$  is surjective, and that each preimage is connected. Finally, for each  $k \leq [d/2]$ , we have a surjective map  $\pi_k : \mathbb{R}/\mathbb{Z} \rightarrow \delta^{-1}\{k\} \subset S$  sending  $\alpha$  to  $\{\alpha, \alpha + k/d\}$ . This map is a 2-to-1 covering map if  $d$  is even and  $k = d/2$ , and 1-to-1 otherwise. Moreover it commutes with any translation in  $\mathbb{R}/\mathbb{Z}$ . Hence every connected component of  $S$  inherits by  $\pi$  a translation structure coming from  $\mathbb{R}/\mathbb{Z}$ .

In short we conclude that:

**Lemma 7.3.** *The space  $S$  has a natural structure of translation manifold of dimension 1. It is compact and has  $[d/2]$  connected components, all of them homeomorphic to  $\mathbb{R}/\mathbb{Z}$ .*

We may now define a translation structure to  $Cb_0$  by way of the following

**Proposition 7.4.** *The set  $Cb_0$  is an open subset of  $S^{d-1}$ , hence admits a natural structure of translation manifold.*

*Proof.* The subset  $Cb_0$  of  $S^{d-1}$  is defined by the two conditions  $\theta_i \neq \theta_j$  and  $\theta_i, \theta_j$  are unlinked for any  $i \neq j$ . Both conditions are clearly open for the Hausdorff topology on compact subsets. Whence  $Cb_0$  is open.  $\square$

*Remark 7.5.* In fact, it can be proven that  $Cb_0$  has finitely many connected components, each homeomorphic to  $\mathbb{R}/\mathbb{Z} \times ]0, 1[^{d-2}$ . The set of connected components is in natural bijection with the set of finite simplicial trees having  $d - 1$  edges labelled with  $\{1, \dots, d - 1\}$ , and such that the set of branches at any vertex is oriented.

The translation manifold  $S^{d-1}$  is compact, hence endowed with a natural positive measure which is of finite mass.

**Definition 7.6.** *We define  $\mu_{Cb_0}$  to be the probability measure proportional to the natural measure on  $S^{d-1}$  and whose support is precisely  $Cb_0$ .*

7.1.2. *The full combinatorial space.* We now describe the set of combinatorics of *all* polynomials of degree  $d$  (with all critical points marked).

**Definition 7.7.** *The set  $Cb$  is the collection of all  $(d - 1)$ -tuples  $(\theta_1, \dots, \theta_{d-1})$  of finite sets in  $\mathbb{R}/\mathbb{Z}$  satisfying the following four conditions:*

- for any fixed  $i$ ,  $\theta_i = \{\alpha_1, \dots, \alpha_{k(i)}\}$  and  $d\alpha_j = d\alpha_1$  for all  $j$ ;
- for any  $i \neq j$ , either  $\theta_i \cap \theta_j = \emptyset$ , or  $\theta_i = \theta_j$ ;
- if  $N$  is the total number of distinct  $\theta_i$ 's, then  $\text{Card } \bigcup_i \theta_i = d + N - 1$ ;
- for any  $i, j$  such that  $\theta_i \cap \theta_j = \emptyset$ , then  $\theta_i$  and  $\theta_j$  are unlinked, that is  $\theta_j$  is contained in a single connected component of  $\mathbb{R}/\mathbb{Z} \setminus \theta_i$ .

A comment on the third item is in order. It may be rephrased as follows: if  $\theta_{i_1}, \dots, \theta_{i_N}$  is a maximal family of disjoint sets in  $(\theta_1, \dots, \theta_{d-1})$ , then  $\sum \text{Card}(\theta_{i_j} - 1) = d - 1$ . This models the fact that our polynomials have exactly  $d - 1$  critical points, counting multiplicities.

We now define a topology on  $\mathbf{Cb}$  (see [K2]). For any collection of open sets  $O_1, \dots, O_{d-1} \subset \mathbb{R}/\mathbb{Z}$ , define  $U(O) = \{\Theta = (\theta_i) \in \mathbf{Cb}, \text{ s.t. } \theta_i \subset O_i\}$ . Any intersection of such sets is of the same form. An open set in  $\mathbf{Cb}$  is by definition an arbitrary union of sets of the form  $U(O)$ .

**Proposition 7.8** (see also [K2, Lemma 3.25]). *The set  $\mathbf{Cb}$  is compact and path connected, and contains  $\mathbf{Cb}_0$  as a dense and open subset.*

*Remark 7.9.* The set  $\mathbf{Cb}$  can be stratified in a natural way such that each stratum is a translation manifold.

**Definition 7.10.** *We let  $\mu_{\mathbf{Cb}}$  be the unique measure which coincides with  $\mu_{\mathbf{Cb}_0}$  on  $\mathbf{Cb}_0$  and which does not charge  $\mathbf{Cb} \setminus \mathbf{Cb}_0$ .*

*Proof.* In order to prove that  $\mathbf{Cb}$  is path connected, it is sufficient to prove that any  $\Theta = (\theta_i)$  can be joined by a continuous path to  $\Theta_\star = (U_d, \dots, U_d)$  where  $U_d = \{0, 1/d, \dots, (d-1)/d\}$ . We prove this by induction on the number  $N$  of distinct  $\theta_i$ 's. When this number equals 1, all  $\theta_i$ 's are equal to the translate of  $U_d$  by some element  $\alpha \in \mathbb{R}/\mathbb{Z}$ . Reducing the parameter of translation to 0 gives us a path joining  $\Theta$  to  $\Theta_\star$ . Now assume the claim has been proven for  $N - 1 \geq 0$ , and suppose  $\Theta$  has  $N$  distinct  $\theta_i$ 's. As before, translate all  $\theta_i$  equal to  $\theta_1$  while leaving the others fixed, until  $\theta_1$  intersects another  $\theta_j \neq \theta_1$ . We conclude by using the inductive hypothesis.

The fact that  $\mathbf{Cb}_0$  is open is clear. A descending induction on the number  $N$  defined before shows that any  $\Theta$  is the limit of a sequence of elements in  $\mathbf{Cb}_0$ . This shows the density of  $\mathbf{Cb}_0$  in  $\mathbf{Cb}$ .

To prove that  $\mathbf{Cb}$  is compact, we let  $\overline{\mathbf{Cb}}_0$  be the closure of  $\mathbf{Cb}_0$  in  $S^{d-1}$  (for the topology induced by the Hausdorff distance). This is a compact space. We claim that there exists a continuous surjective map  $\pi : \overline{\mathbf{Cb}}_0 \rightarrow \mathbf{Cb}$ , which yields the desired result.

Define an equivalence relation  $\sim$  on  $\cup \theta_i$  as follows:  $\alpha \sim \alpha'$  iff there exists a chain  $\theta_{i_1}, \dots, \theta_{i_k}$ , with  $\alpha \in \theta_{i_1}$ ,  $\alpha' \in \theta_{i_k}$ , and for every  $j$ ,  $\theta_{i_j} \cap \theta_{i_{j+1}} \neq \emptyset$ . Define  $\pi : (\theta_1, \dots, \theta_{d-1}) \mapsto (\tilde{\theta}_1, \dots, \tilde{\theta}_{d-1})$ , where  $\tilde{\theta}_i$  is the equivalence class of points contained in  $\theta_i$ . By construction,  $\pi$  is the identity map on  $\mathbf{Cb}_0$ .

We prove that  $\pi(\Theta)$  belongs to  $\mathbf{Cb}$ . The first and second conditions of Definition 7.7 are clearly satisfied. To check the third one, notice that if  $\theta_i$  and  $\theta_j$  belong to  $\overline{\mathbf{Cb}}_0$ ,  $\theta_i \neq \theta_j$ , and  $\theta_i \cap \theta_j \neq \emptyset$ , then  $\text{Card}(\theta_i \cap \theta_j) = 1$ , and use the comment after Definition 7.7.

It remains to prove that the  $\tilde{\theta}_i$  are pairwise unlinked. If, say,  $\tilde{\theta}_1$  and  $\tilde{\theta}_2$  are linked, we get that in the chain  $\theta_{i_1}, \dots, \theta_{i_k}$  constituting  $\tilde{\theta}_1$ , one element is linked with  $\tilde{\theta}_2$ . Applying the same reasoning to the chain constituting  $\tilde{\theta}_2$  implies that two of the subsets in  $\{\theta_1, \dots, \theta_{d-1}\}$  are linked, a contradiction.

This shows that  $\pi : \overline{\mathbf{Cb}}_0 \rightarrow \mathbf{Cb}$  is well-defined. It is continuous by construction, hence its image is compact, and in particular closed. But this image contains  $\mathbf{Cb}_0$  which is dense in  $\mathbf{Cb}$  so  $\pi$  is surjective.  $\square$

**7.1.3. The lamination structure on  $\mathbf{Cb} \times \mathbb{R}_+^*$ .** A locally compact topological space is said to be *laminated by Riemann surfaces* if every point admits a neighborhood  $U_i$  homeomorphic to a product  $\mathbb{D} \times T_i$  where  $\mathbb{D}$  is the unit disk,  $T_i$  is a compact topological space, and such

that the transition function from  $U_i$  to  $U_j$  are continuous and their restriction to complex disks are holomorphic. A *plaque* is a set of the form  $\mathbb{D} \times \{t_i\}$  in some chart. The leaf  $L$  passing through a point  $x$  is the smallest pathwise connected set containing  $x$  and such that if a plaque intersects  $L$  then it is completely included in it. We refer to [Gh] for general facts on laminations.

The abelian group  $\mathbb{R}/\mathbb{Z}$  acts naturally on itself by translation. It hence induces a natural action on  $\mathbf{Cb}$ . If  $\Theta = (\theta_i) \in \mathbf{Cb}$ , with  $\theta_i = \{\alpha_{ij}\}$ , and  $\alpha \in \mathbb{R}/\mathbb{Z}$ , we write  $\Theta + \alpha := (\{\alpha + \alpha_{ij}\}_i)$ . Denote by  $\mathbb{H} = \{z \in \mathbb{C}, \operatorname{Re}(z) > 0\}$  the right half plane. We then have a natural action  $\mathbb{H} \times (\mathbf{Cb} \times \mathbb{R}_+^*) \rightarrow (\mathbf{Cb} \times \mathbb{R}_+^*)$  defined by  $(s + it) \cdot (\Theta, r) := (\Theta + rt, sr)$  —here the group structure on  $\mathbb{H}$  is given by  $(s_1 + it_1) \star (s_2 + it_2) = (s_1 + it_1)s_2 + it_2$ .

**Proposition 7.11.** *There exists a unique structure of lamination by Riemann surfaces on  $\mathbf{Cb} \times \mathbb{R}_+^*$  such that for any fixed  $(\Theta, r) \in \mathbf{Cb} \times \mathbb{R}_+^*$ , the map  $\mathbb{H} \ni u \mapsto u \cdot (\Theta, r)$  is holomorphic. Moreover, all complex leaves of the lamination are analytically diffeomorphic to the punctured unit disk  $\mathbb{D}^*$ .*

*Proof.* Pick  $(\Theta_\star, r_\star) \in \mathbf{Cb} \times \mathbb{R}_+^*$ , with  $\Theta_\star = (\theta_{i,\star})$  and define  $\mathsf{T}_\star$  as the set  $\{\Theta = (\theta_i) \in \mathbf{Cb}, \text{ s.t. } M_d(\theta_1) = M_d(\theta_{1,\star})\}$ , where  $M_d(\alpha) := d\alpha$ . This is a compact set.

For each  $i$ , choose an open set  $O_i \subset \mathbb{R}/\mathbb{Z}$  containing  $\theta_{i,\star}$  and such that any connected component of  $O_i$  has length  $< 1/d^2$  and contains exactly one element of  $\theta_{i,\star}$ . Define  $U_\star := U(O) = \{\Theta = (\theta_i), \theta_i \subset O_i\}$ . This is an open set in  $\mathbf{Cb}$ . Now define the map  $\pi_\star : U \times \mathbb{R}_+^* \rightarrow \mathsf{T}_\star \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}_+^*$  as follows. If  $\Theta$  belongs to  $U_\star$ , then  $\theta_1 \subset O_1$  so  $M_d(\theta_1)$  is a point at distance  $< 1/d$  of  $M_d(\theta_{1,\star})$ . Thus there exists a unique  $\alpha \in \mathbb{R}/\mathbb{Z}$  at distance  $< 1/d$  of 1 such that  $M_d(\alpha + \theta_1) = M_d(\theta_{1,\star})$ , i.e.  $\alpha := M_d(\theta_{1,\star} - \theta_1)/d$ . We set  $\pi(\Theta, r) = (\Theta + \alpha, -\alpha, r)$ . This map is clearly continuous, and injective. Its inverse is given by  $(\Theta, \alpha, r) \mapsto (\Theta + \alpha, r)$ , so the image of  $\pi_\star$  is an open set in  $\mathsf{T}_\star \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}_+^*$ . By postcomposing with the map  $(\Theta, \alpha, r) \mapsto (\Theta, \exp(-2i\pi(\alpha + ir)))$ , we get a map that we again denote by  $\pi_\star$ , which is defined on  $U_\star$  with values in  $\mathsf{T}_\star \times \mathbb{C} \setminus \overline{\mathbb{D}}$ . By definition,  $\pi_\star : U_\star \rightarrow \mathsf{T}_\star \times \mathbb{C} \setminus \overline{\mathbb{D}}$  is a chart for the lamination structure.

We now check the compatibility of this collection of charts. Choose two charts  $\pi_\star, \pi_\bullet$  centered at  $(\Theta_\star, r_\star), (\Theta_\bullet, r_\bullet)$ , and suppose that their domains of definition have non trivial intersection. Then the composition  $\pi_\bullet \circ \pi_\star^{-1}$  is of the form  $(\Theta, \alpha, r) \mapsto (\Theta + \xi, \alpha + \xi, r)$ , with  $\xi := M_d(\theta_{1,\bullet} - \theta_{1,\star})/d$ . For a fixed  $(\Theta, r)$  in the transversal  $\mathsf{T}_\star$ , the composition of  $(\alpha, r) \mapsto \pi_\bullet \circ \pi_\star^{-1}(\Theta, \alpha, r)$  with the projection onto  $\mathbb{R}/\mathbb{Z} \times \mathbb{R}_+^*$  is a translation of angle  $\xi$  in  $\mathbb{R}/\mathbb{Z}$ . Equivalently, it is a complex rotation of angle  $\exp(2i\pi\xi)$  in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ , and is thus holomorphic. This proves that these charts patch together defining a structure of lamination by Riemann surfaces on  $\mathbf{Cb} \times \mathbb{R}_+^*$ .

For fixed  $(\Theta_\star, r_\star)$ , the composition of  $u = (s + it) \in \mathbb{H} \mapsto \pi_\star(u \cdot (\Theta, r))$  with the projection onto the last two factors induces a map  $\mathbb{H} \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$  which is equal to  $u \mapsto \exp(2\pi r_\star u)$  and is clearly holomorphic. One checks that this map is surjective onto the complex leaf of the lamination passing through  $(\Theta_\star, r_\star)$ . The uniqueness of the structure of lamination follows from this remark.

Finally suppose that  $u \cdot (\Theta_\star, r_\star) = u' \cdot (\Theta_\star, r_\star)$  for  $u = s + it$  and  $u' = s' + it'$ . Then  $s = s'$ , and  $rt + \theta_{1,\star} = irt' + \theta_{1,\star}$  as sets. The latter condition is equivalent to  $t - t'$  being congruent to an integer  $k$  depending only on the configuration of  $\theta_{1,\star}$ . The quotient of  $\mathbb{H}$  by  $u \mapsto u + ik$  is the punctured unit disk, which concludes the proof.  $\square$

**7.2. The Goldberg map.** We now explain how the sets  $\mathbf{Cb}_0$  and  $\mathbf{Cb}$  are connected with the parameter space of polynomials.

**7.2.1. Definition.** For any  $r > 0$ , let  $\mathcal{G}(r)$  be the set of polynomials  $P$  of degree  $d$  for which all critical points  $c$  of  $P$  satisfy  $g_P(c) = r$ . We shall see that  $\mathbf{Cb}$  and  $\mathcal{G}(r)$  are closely related. We first recall some basic facts about Böttcher coordinates. For any polynomial  $P$  there exists a unique holomorphic map  $\varphi_P$ , which is defined on  $\{g_P > G(P)\}$ , tangent to the identity at infinity, and such that  $\varphi_P \circ P = \varphi_P^d$ . We call  $\varphi_P$  the Böttcher map. It further satisfies  $g_P = \log |\varphi_P|$  and varies holomorphically with  $P$ . For  $\alpha \in \mathbb{R}/\mathbb{Z}$ , the path  $r \mapsto \varphi_P^{-1}(re^{2i\pi\alpha})$  is called the external ray associated to the angle  $\alpha$ . When  $G(P) > 0$ ,  $\varphi_P^{-1}(re^{2i\pi\alpha})$  tends to a well defined limit point as  $r \rightarrow G(P)$ . We then say that  $\alpha$  is an *external argument* of this point. External rays coincide with gradient lines of  $g_P$ .

The basic proposition is the following.

**Proposition 7.12.** *There exists a unique continuous map  $\Phi_g : \mathbf{Cb} \times \mathbb{R}_+^* \rightarrow \mathcal{P}_d$ ,  $\Phi_g(\Theta, r) = (P(\Theta, r), c_i(\Theta, r))$  such that the following hold.*

- (1) *For each  $i$ , the set of external arguments the critical point of  $c_i$  is  $\theta_i$ , and  $g_{P(\Theta, r)}(c_i) = r$ .*
- (2) *The map  $\Phi_g(\cdot, r)$  is a homeomorphism from  $\mathbf{Cb}$  onto  $\mathcal{G}(r)$ . Moreover,  $\Phi_g$  restricts to a homeomorphism from  $\mathbf{Cb}_0$  onto the subset of  $\mathcal{G}(r)$  of polynomials, all critical points of which are simple.*

*Proof.* Pick  $(P, c) \in \mathcal{G}(r)$ . Each critical point  $c_i$  belongs to the closure of the domain of definition of  $\varphi_P$ , so we may look at the set  $\theta_i := \{\alpha_1^i, \dots, \alpha_k^i\}$  of external arguments of  $c_i$ . We get a collection of finite sets  $\Theta(P, c) = (\theta_i)$ . It is not difficult to check that they satisfy all conditions of Definition 7.7. We thus obtain a continuous map  $\Psi : \mathcal{G}(r) \mapsto \mathbf{Cb} \times \mathbb{R}_+^*$ ,  $\Psi(P, c) = (\Theta(P, c), g_P(c))$ . By [Go, Proposition 3.8],  $\Psi$  is surjective, and by [K2, Lemma 3.22], it is injective. As  $\mathcal{G}(r)$  is compact,  $\Psi$  is a homeomorphism onto  $\mathbf{Cb} \times \{r\}$ , and we denote by  $\Phi_g : \mathbf{Cb} \times \{r\} \rightarrow \mathcal{G}(r)$  its inverse. By construction, (1) is satisfied. Furthermore, for any  $(P, c) \in \mathcal{G}(r)$ , the polynomial  $P$  has a simple critical point at  $c_i$  iff  $c_i$  has exactly two external arguments. This implies that the image of  $\mathbf{Cb}_0$  by  $\Phi_g$  is the set of polynomials with only simple critical points, and proves (2).  $\square$

**Proposition 7.13.** *The restriction of  $\Phi_g$  to any leaf of the underlying lamination of  $\mathbf{Cb} \times \mathbb{R}_+^*$  is holomorphic. In particular,  $\Phi_g$  induces an embedding of the natural lamination of  $\mathbf{Cb} \times \mathbb{R}_+^*$  into  $\mathcal{P}_d \setminus \mathcal{C}$ .*

**Remark 7.14.** For a fixed  $(\Theta, r)$ , the image under  $\Phi_g$  of the leaf  $\mathbb{H} \cdot (\Theta, r)$  coincides with the wringing curve of  $\Phi_g(\Theta, r)$  as defined by Branner-Hubbard in [BrH1].

The proposition is based on the following remark. Suppose  $P \in \mathcal{G}(r)$ , then  $g_P(c_i) = r$ , so that we have  $g_P(P(c_i)) = dr > G(P)$ . Hence we may consider the holomorphic maps  $\varphi_i(P, c) := \varphi_P(P(c_i))$  in a neighborhood of  $P$ . This defines a holomorphic map  $\varphi := (\varphi_1, \dots, \varphi_{d-1})$  in the neighborhood of  $\mathcal{G}(r)$  taking its values in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ . We have

**Proposition 7.15.** *The set of points where the differential of  $\varphi$  is not locally invertible is a complex hypersurface  $H$ , such that for all  $r > 0$ ,  $H \cap \mathcal{G}(r)$  has no interior points in  $\mathcal{G}(r)$ .*

*Proof of Proposition 7.13.* Pick  $(\Theta, r) \in \mathbf{Cb}$ . We claim that the restriction of  $\Phi_g$  to the complex leaf passing through  $(\Theta, r)$  is holomorphic. By Proposition 7.11, this complex leaf is  $\mathbb{H} \cdot (\Theta, r)$ , and a direct computation shows that  $\mathbb{H} \ni u \mapsto (\varphi \circ \Phi_g)[u \cdot (\Theta, r)] = \exp(dr(u -$

$1))\varphi(P)$  if  $\Phi_g(\Theta, r) = (P, c)$ . This map is clearly holomorphic as a function of  $u$ . For any  $(P, c)$  outside  $H$ ,  $\varphi$  gives local holomorphic coordinates, so  $u \mapsto \Phi_g(u \cdot (\Theta, r))$  is holomorphic for  $(\Theta, r)$  in the dense subset  $\mathbf{Cb}_0 \cap \Phi_g^{-1}(\mathcal{P}_d \setminus H)$ . For general  $(P, c)$ , we conclude by continuity.  $\square$

*Proof of Proposition 7.15.* First note that  $\mathcal{G}(r) = \bigcap_i \{\log |\varphi_i| = dr\}$  is a (possibly singular) real-analytic subset of  $\mathcal{P}_d$ . Although we do not strictly need it, we also note that  $\mathcal{G}(r)$  has pure real dimension  $d-1$ , as it is homeomorphic to  $\mathbf{Cb}$  by  $\Phi$ . Consider the continuous map  $\psi$ , sending  $(\theta_1, \dots, \theta_{d-1}) \in \mathbf{Cb}$  to  $[\exp(dr + iM_d(\theta_1)), \dots, \exp(dr + iM_d(\theta_{d-1}))] \in (e^{dr}\mathbb{S}^1)^{d-1}$  (with  $M_d(\alpha) = d \times \alpha$ ). By construction, we have  $\varphi = \psi \circ \Phi^{-1}$ . As  $\psi$  is open, the restriction map  $\varphi|_{\mathcal{G}(r)}$  is open. Sard's Theorem implies that its differential has generically maximal rank. In other words, if  $(P, c)$  belongs to some full measure subset in  $\mathcal{G}(r)$ , the differential of  $\varphi$  restricted to the tangent space of  $\mathcal{G}(r)$  has (real) rank equal to the (real) dimension of  $\varphi(\mathcal{G}(r)) = (e^{dr}\mathbb{S}^1)^{d-1}$  which is  $d-1$ . Moreover, the image of  $d\varphi_{P,c}$  contains the tangent space at  $(e^{dr}\mathbb{S}^1)^{d-1}$  which is totally real. But  $d\varphi_{P,c}$  is a complex linear map, so its image necessarily contains a complex vector space of dimension  $d-1$ . This proves that  $\varphi$  is locally invertible at generic points of  $\mathcal{G}(r)$ . In particular  $H = \{\det d\varphi = 0\}$  is a complex hypersurface whose intersection with  $\mathcal{G}(r)$  has no interior points, and  $d\varphi$  is invertible outside  $H$ .  $\square$

**7.2.2. Kiwi's continuity property for Misiurewicz combinatorics.** Fix  $\Theta \in \mathbf{Cb}$ . The (*stretching*) ray associated to  $\Theta$  is the set  $\{P_r = P(\Theta, r), r > 0\}$ . When  $r \rightarrow 0$ , then  $G(P_r)$  converges to 0 so that any polynomial in the cluster set of  $\{P_r\}$  belongs to the connectedness locus. It is a very delicate problem to describe this cluster set in general. We say that a ray *lands* if this cluster set is a single point.

**Definition 7.16.** A combinatorics  $\Theta = (\theta_i)$  is said to be of Misiurewicz type, if any  $\alpha \in \bigcup \theta_i$  is strictly preperiodic under the map  $z \mapsto dz$ . We denote by  $\mathbf{Cb}_{\text{mis}}$  the set of Misiurewicz combinatorics.

**Proposition 7.17.** The set  $\mathbf{Cb}_{\text{mis}}$  is dense in  $\mathbf{Cb}$ .

*Proof.* The set of periodic orbits of  $z \mapsto dz$  is dense in  $\mathbb{R}/\mathbb{Z}$ . Pick any finite set  $\theta = \{\alpha_1, \dots, \alpha_j\} \subset \mathbb{R}/\mathbb{Z}$  such that  $\alpha := d\alpha_1 = \dots = d\alpha_j$ . One can then find a periodic point  $\alpha_\star$  arbitrarily close to  $\alpha$  whose orbit does not intersect  $\theta$ . Let  $\theta' = \theta + \frac{1}{d}(\alpha_\star - \alpha)$ . This is a finite set very close to  $\theta$ , and strictly preperiodic.

Now let  $\Theta = (\theta_i) \in \mathbf{Cb}$ . For each  $i$ , consider the set  $I_i$  of all indices  $j$  such that  $\theta_j = \theta_i$ . The preceding argument shows that we may translate all  $\{\theta_j\}_{j \in I_i}$  at the same time so that they become strictly preperiodic. Doing the same for each  $i$ , we get a combinatorics  $\Theta'$  which is of Misiurewicz type and arbitrarily close to  $\Theta$ .  $\square$

We now state without proof the following deep continuity result, which is a combination of the results of [K2] and [BFH], see [K2, Corollary 5.3].

**Theorem 7.18.** The map  $\Phi_g$  extends continuously to  $\mathbf{Cb}_{\text{mis}} \times \{0\}$ . The extended map  $\Phi_g$  induces a bijection from  $\mathbf{Cb}_{\text{mis}} \times \{0\}$  onto the subset of Misiurewicz polynomials.

Note that the continuity statement is particularly strong. It means that for any  $\Theta \in \mathbf{Cb}_{\text{mis}}$ , the sequence  $P(\Theta, r)$  converges when  $r \rightarrow 0$ , and that the polynomial is Misiurewicz. It also means that for any sequence  $(\Theta_n, r_n) \rightarrow (\Theta, 0)$ ,  $P(\Theta_n, r_n)$  converges to  $P(\Theta, 0)$ .

**7.2.3. Measurable landing.** We now prove a statement somewhat dual to the previous theorem. Note that this result was already observed in [BMS, Theorem B].

**Proposition 7.19.** *For any  $\Theta$  and for almost every  $t > 0$ , the map  $\Phi_g((s + it) \cdot (\Theta, 1))$  has a limit when  $s \rightarrow 0$ . In particular, for  $\mu_{\text{Cb}}$ -almost every  $\Theta \in \text{Cb}$ , the map  $r \mapsto \Phi_g(\Theta, r)$  admits a limit when  $r \rightarrow 0$ .*

*Proof.* Let us prove the first statement. Fix  $\Theta \in \text{Cb}$ . By Proposition 7.12, the map  $\mathbb{H} \ni u = (s + it) \mapsto \Phi_g((s + it) \cdot (\Theta, 1)) = \Phi_g(\Theta + t, s) \in \mathcal{P}_d$  is holomorphic. For simplicity, we lift this map to the ramified cover  $\pi : \mathbb{C}^{d-1} \rightarrow \mathcal{P}_d$  that we used in Section 5. We get a map  $\varphi : \mathbb{H} \rightarrow \mathbb{C}^{d-1}$  such that  $G(\varphi(u)) = \text{Re}(u) \times G(\Phi_g(\Theta, 1))$ . In particular, the restriction of  $\varphi$  to any square  $\{0 < \text{Re}(u) < 1, |\text{Im}(u)| \leq R\}$  with  $R > 0$  is a holomorphic and bounded function. By Fatou's theorem (see e.g. [Mi2, Lemma 15.1])  $\lim_{s \rightarrow 0} \varphi(s + it)$  exists for almost every  $|t| \leq R$ . We conclude by letting  $R \rightarrow \infty$ .

For the second statement, let  $\mathcal{B}$  be the set of  $\Theta \in \text{Cb}$  such that  $\Phi_g(\Theta, r)$  admits a limit when  $r \rightarrow 0$ . We have proved that for any  $\Theta$ , the set of  $t > 0$  for which  $\Theta + t$  belongs to  $\mathcal{B}$  has full (Lebesgue) measure in  $\mathbb{R}$ . Now recall that  $\mu_{\text{Cb}}$  puts full measure on  $\text{Cb}_0$ , that  $\text{Cb}_0$  is an open set of  $(\mathbb{R}/\mathbb{Z})^{d-1}$  and that the measure  $\mu_{\text{Cb}}$  is the restriction of the natural Haar measure on  $(\mathbb{R}/\mathbb{Z})^{d-1}$ . By Fubini's Theorem, we conclude that  $\mu_r(\mathcal{B}) = 1$ .  $\square$

### 7.3. Landing measure.

Our main result is

**Theorem 7.20.** *For any  $r > 0$ , the Monge-Ampère measure associated to  $\max\{G, r\}$  is equal to the image of  $\mu_{\text{Cb}}$  under the map  $\Theta \mapsto \Phi_g(\Theta, r)$ .*

As a corollary, we can give a proof of Theorem 8 stated in the introduction.

*Proof of Theorem 8.* By Proposition 7.19, the limit  $e(\Theta) := \lim_{r \rightarrow 0} \Phi_g(\Theta, r)$  exists for  $\mu_{\text{Cb}}$ -almost every  $\Theta$ . This yields a measurable map  $e : \text{Cb} \rightarrow \partial\mathcal{C}$ . Now for any  $r > 0$ ,  $\Phi_g(\cdot, r)_* \mu_{\text{Cb}} = (dd^c)^{d-1} \max\{G, r\}$  by Theorem 7.20. As  $\max\{G, r\}$  decreases to the continuous psh function  $G$  when  $r$  decreases to 0, we have  $\lim_{r \rightarrow 0} (dd^c)^{d-1} \max\{G, r\} = \mu_{\text{bif}}$ , see [BT1]. On the other hand  $\Phi_g(\cdot, r)$  converges measurably to  $e$ . Whence  $e_* \mu_{\text{Cb}} = \mu_{\text{bif}}$ .  $\square$

*Proof of Theorem 7.20.* Look at the following commutative diagram

$$\begin{array}{ccc} \text{Cb} & & \\ \downarrow \Phi & \searrow \psi & \\ \mathcal{G}(r) & \xrightarrow{\varphi} & (e^{dr}\mathbb{S}^1)^{d-1} \end{array}$$

where  $\Phi = \Phi_g(\cdot, r)$ ,  $\varphi(P, c) = (\varphi_P P(c_i))$  and  $\psi(\theta_1, \dots, \theta_{d-1}) := (e^{dr+iM_d(\theta_1)}, \dots, e^{dr+iM_d(\theta_{d-1})})$  as in the proof of Proposition 7.15. By construction,  $\varphi \circ \Phi = \psi$ , and recall that by Proposition 7.12, the map  $\Phi$  is a homeomorphism. Let  $\mu_r = (dd^c)^{d-1} \max\{G, r\}$ . We show that  $\Phi^* \mu_r = \mu_{\text{Cb}}$  (where  $\Phi^*$  stands for  $(\Phi^{-1})_*$ ). Denote by  $d\lambda$  the Haar measure on the  $(d-1)$  real dimensional torus  $(e^{dr}\mathbb{S}^1)^{d-1}$ .

**Lemma 7.21.** *There exists an open set  $G \subset \Phi(\text{Cb}_0)$ , such that  $\mu_r(G) = \mu_{\text{Cb}}(\Phi^{-1}(G)) = 1$ , and every point in  $G$  admits an open neighborhood  $U$  on which  $\varphi_* \mu_r|_U$  coincides with  $d^{1-d} \times d\lambda|_{\varphi(U)}$ .*

Pick  $(P, c) \in G$ . Take  $U$  as in the lemma so that  $\varphi_*\mu_r|_U = d^{1-d} \times d\lambda|_{\varphi(U)}$ . As  $G \subset \Phi(\mathbf{Cb}_0)$ , the polynomial  $P$  has only simple critical points, and we may assume that this property is satisfied all over  $U$ . Let  $V$  be the inverse image of  $U \cap \mathcal{G}(r)$  by  $\Phi$ . By Proposition 7.12, it is contained in  $\mathbf{Cb}_0$ . Now note that  $\psi : \mathbf{Cb}_0 \rightarrow (e^{dr}\mathbb{S}^1)^{d-1}$  preserves the structure of translation manifolds of both spaces, so  $\psi_*\mu_{\mathbf{Cb}}$  is proportional to  $d\lambda$ . In particular,  $\psi_*\mu_{\mathbf{Cb}}|_V = t \times d\lambda$  where  $t^{-1} = \lambda(\psi(\mathbf{Cb}_0))$  does not depend on  $V$ . We infer that  $\Phi^*\mu_r = t' \times \mu_{\mathbf{Cb}}$  on  $V$  hence on  $\Phi^{-1}(G)$  with  $t' = d^{1-d}/t$ . But  $G$  has full  $\mu_r$ -measure, and  $\Phi^{-1}(G)$  has full  $\mu_{\mathbf{Cb}}$ -measure, so  $\Phi^*\mu_r = t' \times \mu_{\mathbf{Cb}}$ . Both  $\mu_r$  and  $\mu_{\mathbf{Cb}}$  being probability measures,  $\Phi^*\mu_r = \mu_{\mathbf{Cb}}$ , as required.  $\square$

*Proof of Lemma 7.21.* The same proof as for Lemma 6.15 gives in a neighborhood  $\mathcal{N}$  of  $\mathcal{G}(r)$ ,

$$\begin{aligned} \mu_r = (dd^c)^{d-1} \max\{G, r\} &= dd^c \max\{g_P(c_1), r\} \wedge \cdots \wedge dd^c \max\{g_P(c_{d-1}), r\} \\ &= dd^c \max\{d^{-1} \log |\varphi_1|, r\} \wedge \cdots \wedge dd^c \max\{d^{-1} \log |\varphi_{d-1}|, r\} \end{aligned}$$

Apply Proposition 7.15, and set  $G := \{(P, c) \in \mathcal{N} \setminus H, \text{ s.t. } P \text{ has only simple critical points}\}$ . For any point in  $G$ , there exists an open neighborhood such that the mapping  $\varphi$  is a holomorphic diffeomorphism from  $U$  onto its image, so

$$\begin{aligned} \mu_r|_U &= \varphi|_U^* (dd^c \max\{d^{-1} \log |z_1|, r\} \wedge \cdots \wedge dd^c \max\{d^{-1} \log |z_{d-1}|, r\}) \\ &= d^{1-d} \varphi|_U^* (dd^c \max\{\log |z_1|, r^d\} \wedge \cdots \wedge dd^c \max\{\log |z_{d-1}|, r^d\}) \end{aligned}$$

Now for any real number  $\rho > 0$ , the measure  $dd^c \max\{\log |z_1|, \rho\} \wedge \cdots \wedge dd^c \max\{\log |z_{d-1}|, \rho\}$  in  $\mathbb{C}^{d-1}$  is a probability measure supported on  $(e^\rho \mathbb{S}^1)^{d-1}$  and invariant under the subgroups of rotations  $\simeq (\mathbb{S}^1)^{d-1}$ . So it equals the Haar measure  $d\lambda$ .

In order to conclude, we need to show that  $\mu_r(G) = \mu_{\mathbf{Cb}}(\Phi^{-1}(G)) = 1$ . First  $\mu_r$  does not charge complex analytic sets, so  $\mu_r(G) = \mu_r(\mathcal{N}) = 1$ . Now  $\Phi^{-1}(H) \subset \psi^{-1}\varphi(H)$ , and  $\varphi(H)$  is a complex analytic set in  $\mathbb{C}^{d-1}$ . Its intersection with the torus  $(e^{dr}\mathbb{S}^1)^{d-1}$  is hence of zero Haar measure. Finally  $\mu_{\mathbf{Cb}}\Phi^{-1}(H) \leq \mu_{\mathbf{Cb}}\psi^{-1}\varphi(H) = (\psi_*\mu_{\mathbf{Cb}})\varphi(H) = t \times \lambda\varphi(H) = 0$ .  $\square$

**7.4. A connectedness property.** The next proposition gives a more accurate picture of the geometry of the boundary of the connectedness locus. When  $d = 2$ , this is the statement that the Mandelbrot set is connected. Let  $\mathcal{L}$  be the subset of the shift locus consisting of the polynomials such that  $g_P(c_1) = \cdots = g_P(c_{d-1})$ , that is,  $\mathcal{L} = \Phi_g(\mathbf{Cb} \times \mathbb{R}_+^*)$ . Note that  $\text{Supp}(\mu_{\text{bif}}) \subset \mathcal{C} \cap \overline{\mathcal{L}}$ .

**Proposition 7.22.** *The set  $\mathcal{C} \cap \overline{\mathcal{L}}$  is connected.*

*Proof.* For  $r > 0$ , recall the set  $\mathcal{G}(r)$  of polynomials for which all critical points satisfy  $g_P(c) = r$ . It is homeomorphic to  $\mathbf{Cb}$ , hence connected. Let now  $\mathcal{G}(\leq r)$  be the union of  $\mathcal{G}(s)$  for  $0 < s \leq r$ . Being homeomorphic to  $\mathbf{Cb} \times ]0, r]$  it is connected. Hence  $\overline{\mathcal{G}(\leq r)}$  is compact and connected. Finally

$$\bigcap_{r>0} \overline{\mathcal{G}(\leq r)} = \overline{\mathcal{L}} \cap \mathcal{C}$$

is connected.  $\square$

**7.5. Proof of Theorem 9.** We prove that any Misiurewicz parameter  $(P, c)$  lies in the support of  $\mu_{\text{bif}}$ . Apply Theorem 7.18, and pick  $\Theta_* \in \mathbf{Cb}_{\text{mis}}$  such that  $\Phi_g(\Theta_*, 0) = (P, c)$ . Apply now Theorem 7.20:  $e_*\mu_{\mathbf{Cb}} = \mu_{\text{bif}}$ . In particular  $e(\Theta)$  is well defined and belongs to the support of  $\mu_{\text{bif}}$  for  $\mu_{\mathbf{Cb}}$ -almost every  $\Theta$ . Now,  $\mu_{\mathbf{Cb}}$  has full support in  $\mathbf{Cb}$ . Thus we can find a sequence  $\Theta_k$  converging to  $\Theta_*$  and such that  $e(\Theta_k)$  lies in the support of  $\mu_{\text{bif}}$ . For each  $k$ , pick

$\varepsilon_k > 0$  small enough such that the distance between  $\Phi_g(\Theta_k, \varepsilon_k)$  and  $e(\Theta_k)$  is less than  $1/k$ . In particular, the distance between  $\Phi_g(\Theta_k, \varepsilon_k)$  and  $\text{Supp } \mu_{\text{bif}}$  tends to 0 when  $k \rightarrow \infty$ . By the Continuity Theorem 7.18, we have  $\Phi_g(\Theta_k, \varepsilon_k) \rightarrow (P, c)$ . Whence  $(P, c)$  is in the support of the bifurcation measure.

The reverse inclusion was already proved in Corollary 6.  $\square$

**7.6. Topological Collet-Eckmann property.** In this section we give the proof of Theorem 10 (and Corollary 11). This is an adaptation of the proof of Smirnov [Sm]. Although Smirnov worked only with unicritical polynomials, most of his arguments remains valid in the multi-critical case as well, see [Sm, Remark 1] and [BMS, p. 348]. The other two ingredients in the proof are the work of Kiwi on the combinatorics of multi-critical polynomials, [K1] and our landing Theorem 8.

*Proof of Theorem 10.* Recall that the TCE condition reads as follows: the polynomial  $P$  satisfies the TCE condition if for some  $A \geq 1$  there exists constants  $M > 1$  and  $r > 0$  such that for every  $x \in J_P$  there is an increasing sequence  $(n_j)$  with  $n_j \leq Aj$  such that for every  $j$ ,

$$(13) \quad \# \left\{ i, 0 \leq i < n_j, \text{Comp}_{f^i(x)} f^{-(n_j-i)} B(f^{n_j}(x), r) \cap \text{Crit} \neq \emptyset \right\} \leq M,$$

where Crit denotes the critical set, and  $\text{Comp}_x X$  is the connected component of the set  $X$  containing  $x$ . More precisely we refer to (13) as the  $\text{TCE}(M, A, r)$  condition. When  $P$  is a polynomial with marked critical points  $(c_1, \dots, c_{d-1})$ , we say that  $P$  satisfies the  $\text{TCE}_k(M, A, r)$  condition if (13) holds with  $c_k$  instead of Crit. It is clear that if  $P$  fails the  $\text{TCE}(M, A, r)$  condition, it fails the  $\text{TCE}_k(\frac{M}{d-1}, A, r)$  condition for some  $k$ .

We now introduce the following subset  $\text{Cb}_1 \subset \text{Cb}$  of combinatorics  $\Theta$  satisfying the following three conditions:

- the ray  $\{\Phi_g(\Theta, r)\}_{r>0}$  lands at a polynomial with marked critical points  $e(\Theta) := (P, c)$ .
- $P$  has only simple critical points (i.e.  $\Theta \in \text{Cb}_0$ ), and none of them is preperiodic.
- $P$  has only repelling cycles; in particular  $K_P = J_P$ .

Note that the first condition is  $\mu_{\text{Cb}}$ -generic by Theorem 8; the second is also generic as  $\mu_{\text{bif}}$  does not charge hypersurfaces. Finally [K2, Theorem 1], and [BMS, Lemma 5], implies the genericity of the last condition too. So  $\text{Cb}_1$  has *full measure* in  $\text{Cb}$ .

We will prove that the set of combinatorics  $\Theta \in \text{Cb}_1$  for which the associated polynomial  $P$  violates the  $\text{TCE}_k(\frac{M}{d-1}, A, r)$  condition for some  $A$  and every  $M, r$  has zero  $\mu_{\text{bif}}$  measure. Without loss of generality, assume  $k = 1$ .

We use external rays in dynamical plane to understand the recurrence property of critical points on the Julia set. As external rays do not land in general, we are led to work with *fibers*, originally introduced by Schleicher [Sch].

Recall that external rays with rational angles always land at preperiodic points [DH]. By definition, two points  $\xi, \zeta \in J_P$  do not belong to the same fiber if there exist external rays  $R_s$  and  $R_t$ , with rational angles, landing at a common point  $z$ , such that  $R_s \cup \{z\} \cup R_t$  separates  $\xi$  from  $\zeta$ . We say that  $\theta$  is an *external argument* of  $\zeta \in J_P$  if  $\overline{R_\theta} \cap J_P \subset \text{Fiber}(\zeta)$ . It follows from the work of Kiwi [K1, Theorem 3] that when  $P$  has only repelling cycles, every  $\zeta \in J_P$  has a nonempty finite set of external arguments.

The following lemma is the classical connection between external arguments in dynamical and parameter spaces.

**Lemma 7.23.** *Let  $\Theta = (\theta_1, \dots, \theta_{d-1}) \in \mathsf{Cb}_1$ , with  $\theta_1 = \{\alpha, \alpha'\} \in S$ . Write  $e(\Theta) = (P, c_1, \dots, c_{d-1})$ . Then  $\overline{R_\alpha} \cap J_P$  (resp.  $\overline{R_{\alpha'}} \cap J_P$ ) is contained in the fiber of  $c_1$ , that is,  $\alpha$  and  $\alpha'$  are external arguments of  $c_1$ .*

*Proof.* Assume that some accumulation point  $\zeta$  of  $R_\alpha$  does not belong to the fiber of  $c_1$ . Then there exists a pair of rational rays  $R_s$  and  $R_t$  landing at  $z \in J_P$  separating  $\zeta$  from  $c_1$ . The point  $z$  is necessarily prerepelling and not precritical, because  $P$  has no preperiodic critical point and all cycles repelling. Now, if we perturb  $P$  in the parameter space of polynomials,  $z$  admits a continuation as a prerepelling point and for the perturbed map, the rays  $R_s$  and  $R_t$  still land at the continuation of  $z$  (see e.g. [K2, Lemma 5.2]). But in the parameter ray associated to  $\Theta$ ,  $\theta_1 = \{\alpha, \alpha'\}$  is the set of external arguments of  $c_1$  so  $c_1$  and  $R_\alpha$  are in the same connected component of  $\mathbb{C} \setminus (R_s \cup R_t \cup \{z\})$ , a contradiction.  $\square$

**Lemma 7.24.** *Let  $I_0$  and  $I_1$  be the two connected components of  $\mathbb{R}/\mathbb{Z} \setminus \{\alpha, \alpha'\}$ . For  $\varepsilon = 0, 1$ , define  $J_\varepsilon$  to be the set of points in  $J_P \setminus \text{Fiber}(c_1)$  having an external argument in  $I_\varepsilon$ .*

*Then the three sets  $J_0$ ,  $J_1$  and  $\text{Fiber}(c_1)$  form a partition of  $J_P$ , and all iterates of  $c_1$  lie in  $J_0 \cup J_1$ .*

*Proof.* By [K1, Proposition 3.15],  $\text{Fiber}(c_1)$  is compact, connected and full. Hence  $R_\alpha \cup \text{Fiber}(c_1) \cup R_{\alpha'}$  separates the plane into two connected components,  $U_0$  and  $U_1$ . Define  $J_\varepsilon := U_\varepsilon \cap J_P$ . By construction an external ray with argument in  $I_\varepsilon$  is contained in  $U_\varepsilon$ . So any external angle of a point in  $J_\varepsilon$  belongs to  $I_\varepsilon$ .

Finally, by [K1, Corollary 2.15], the image of a fiber is a fiber, and, by [K1, Lemma 4.4], if  $\text{Fiber}(c_1)$  is periodic, then  $c_1$  itself is periodic. By assumption  $c_1$  is not preperiodic, so all iterates of  $c_1$  lie in  $J_0 \cup J_1$ .  $\square$

We now follow Smirnov's proof. Pick  $\Theta \in \mathsf{Cb}_1$ , and write  $e(\Theta) = (P, c)$ . By the previous lemma,  $c_1$  has a well defined itinerary in the space  $\Sigma_2 := \{0, 1\}^{\mathbb{N}^*}$ . Call this itinerary the *kneading sequence* of  $c_1$ . In this way, we get a map  $\kappa : \mathsf{Cb}_1 \rightarrow \Sigma_2$ , characterized by the condition that  $P^n(c_1) \in U_{\varepsilon_n}$  for all  $n$  with  $\kappa(\Theta) = (\varepsilon_1, \varepsilon_2, \dots)$ .

By repeating [Sm, Section 2], we get that if  $P$  fails the TCE<sub>1</sub>( $M, A, r$ ) condition for some  $A$  and every  $M, r$ , then  $\kappa(P)$  is *Strongly Recurrent*. We refer to [Sm] for a precise definition of this condition. The key fact is that the set SR of strongly recurrent sequences has zero Hausdorff dimension in  $\Sigma_2$ . Here we endow  $\Sigma_2$  with its usual 2-adic metric, that is,  $d(x, y) = 2^{-n}$ , where  $n$  is the smallest integer such that  $x_n \neq y_n$ . Notice that the Hausdorff dimension of  $(\Sigma_2, d)$  is 1. We now rely on the following lemma which is the analogue of [Sm, Proposition 1].

**Lemma 7.25.** *The Hausdorff dimension of the set  $\kappa^{-1}(\text{SR}) \subset \mathsf{Cb}_1$  is not greater than  $(d - 2) + \log(d - 1)/\log d < d - 1 = \dim(\mathsf{Cb}_1)$ .*

The lemma implies that  $\kappa^{-1}(\text{SR}) \cap \mathsf{Cb}_1$  has zero  $\mu_{\mathsf{Cb}}$ -measure, and the proof of the theorem is complete.  $\square$

*Proof of Lemma 7.25.* First note that  $\kappa$  is the composition of the projection  $\pi : \mathsf{Cb}_1 \rightarrow S$  onto the first factor  $\Theta \mapsto \theta_1$ , together with the map  $K : S \rightarrow \Sigma_2$  defined as follows. If  $\theta := \{\alpha, \alpha'\} \in S$ , then set the  $n^{\text{th}}$  term of  $K(\theta)$  to be  $\varepsilon$  iff  $d^n \alpha = d^n \alpha' \in I_\varepsilon$ . We will prove that the Hausdorff dimension of  $K^{-1}(\text{SR})$  is not greater than  $\log(d - 1)/\log d$ , which implies the lemma.

It is enough to restrict to one connected component  $S_0$  of  $S$ . For simplicity we assume that the distance between  $\alpha$  and  $\alpha'$  is  $k/d$  with  $k < d/2$ , so the component  $S_0$  is parameterized by  $\alpha \in \mathbb{R}/\mathbb{Z}$ , and  $\alpha' = \alpha + k/d$ . The remaining case  $k = d/2$  is left to the reader.

To get the dimension estimate, we prove that if  $C_n$  is any cylinder of depth  $n$  in  $\Sigma_2$ , then  $K^{-1}(C_n)$  consists of at most  $C^{st} \times n(d-k)^n$  intervals of length at most  $d^{-n}$  (here  $d-k \geq 2$  since  $d \geq 3$ ). This easily implies that the dimension of  $K^{-1}(SR) \cap S_0$  is at most  $\log(d-k)/\log d$  (compare with [BS, Section 8]).

First it is clear that the  $n^{\text{th}}$  digit of the kneading sequence cannot remain constant if  $\alpha$  gets increased by an amount of  $1/d^n$ : indeed  $d^n\alpha$  turns once around  $\mathbb{R}/\mathbb{Z}$  so it hits  $\alpha$  or  $\alpha'$  which are almost fixed and at distance  $k/d$ . The estimate of the number  $N(n)$  of intervals goes by induction on  $n$ . Let  $M_d$  be the multiplication-by- $d$  map on  $\mathbb{R}/\mathbb{Z}$ . Recall that  $\mathbb{R}/\mathbb{Z} = I_0 \cup I_1 \cup \{\alpha, \alpha'\}$ , and let  $\ell_k/d$  be the length of  $I_k$ ,  $\ell_k \in \{k, d-k\}$ . Let  $I$  be any interval in  $\mathbb{R}/\mathbb{Z}$  of length  $< 1/d$ . There are two cases. Either  $d\alpha \notin I$  and  $M_d^{-1}(I) \cap I_k$  consists of  $\ell_k$  intervals, or  $d\alpha \in I$  and  $M_d^{-1}(I) \cap I_k$  consists of  $\ell_k+1$  intervals. Of course the latter case occurs for at most one of the  $N(n)$  intervals. We get that  $N(n+1) \leq (d-k)(N(n)-1) + (d-k+1) = (d-k)N(n) + 1$ , whence  $N(n) \leq C^{st} \times n(d-k)^n$ .  $\square$

*Proof of Corollary 11.* We already noticed in the proof of the previous theorem that the first assertion is a consequence of [K2, Theorem 1], and [BMS, Lemma 5]. The third one is a combination of [GSm, P2] for the local connectivity statement, and [PR] for the statement on Hausdorff dimension. Lastly, the second item follows from the proof of the theorem. Indeed, if  $P$  satisfies the TCE property, then  $J_P$  is locally connected, so the landing of external rays defines a continuous map  $\mathbb{R}/\mathbb{Z} \rightarrow J_P$ , semiconjugating  $P$  to multiplication by  $d$ . If  $P = e(\Theta)$ , the angles in  $\theta_i$  are external arguments of  $c_i$ , and for generic  $\Theta$ , they have dense orbit on  $\mathbb{R}/\mathbb{Z}$  under multiplication by  $d$ .  $\square$

## 8. THE SPACE OF RATIONAL MAPS

In this last section we indicate how some of the results of Section 6 can be extended to the space of rational maps of degree  $d$  with critical points marked. Since the methods and results are quite similar, the exposition is somewhat sketchy.

Let  $\text{Rat}_d$  be the space of all rational maps of degree  $d \geq 2$ , with  $(2d-2)$  marked critical points, modulo conjugation by Möbius transformations. A point in this space is a  $(2d-1)$ -tuple  $(R, c_i)$  such that  $R$  is a rational map of degree  $d$ , and  $\{c_i\}$  is the set of critical points of  $R$  (counted with repetition according to their multiplicities). This space is a finite ramified cover over the space of rational maps of degree  $d$  up to conjugacy, hence  $\text{Rat}_d$  is a quasiprojective algebraic variety, see [Mi1, Si]. Note that in general it has singularities. Denote by  $T_k$  the positive closed  $(1,1)$  current describing the bifurcation of the marked critical point  $c_k$  as defined in Section 3.

It is a consequence of [McM2, Theorem 2.2], that the set of critically finite maps is the union of countably many points, together with a quasi-projective curve  $\mathcal{L}$ . This curve appears only in degrees which are squares of integers, and consists in the set of Lattès maps whose lift to some elliptic curve is the multiplication by  $\sqrt{d}$ .

**Theorem 8.1.** *For each  $l < 2d-2$ , denote by  $\Gamma_l$  an irreducible component of the quasiprojective subvariety of  $\text{Rat}_d \setminus \mathcal{L}$  consisting of rational maps such that the critical points  $c_1, \dots, c_l$  are preperiodic.*

*The codimension of  $\Gamma_l$  is precisely  $l$ ; the restriction of the current  $T_{l+1}|_{\Gamma_l}$  is a non-zero positive closed  $(1,1)$  current; and for any sequence of integers  $k(n) < n$ , we have*

$$(14) \quad \lim_{n \rightarrow \infty} (d^n + d^{k(n)})^{-1} [\text{Per}_{l+1}(n, k(n)) \cap \Gamma_l] = T_{l+1}|_{\Gamma_l},$$

where  $\text{Per}_{l+1}(n, k(n))$  denotes the set  $\left\{ R_\lambda^n c_{l+1}(\lambda) = R_\lambda^{k(n)} c_{l+1}(\lambda) \right\}$ .

Another version of this result has recently been obtained by Bassanelli-Berteloot, see [BB2]. Their method is different although it bears the same analytic techniques: it is based on the description of the bifurcation current in terms of the Lyapunov function on the parameter space.

*Proof.* For the proof we fix a decreasing sequence of components  $\Gamma_1 \supset \Gamma_2 \cdots \supset \Gamma_{2d-2}$  satisfying the conditions of the theorem.

Being defined by  $l$  equations,  $\text{codim } \Gamma_l \geq l$  for all  $l$ . In  $\text{Rat}_d \setminus \mathcal{L}$ , the set of critically finite rational maps is countable, so any of its irreducible component is a point. This shows that  $\Gamma_{2d-2}$  is necessarily a point. By induction, we conclude that  $\text{codim } \Gamma_l = l$ . Moreover  $\Gamma_{l+1}$  has codimension 1 in  $\Gamma_l$ , so the family of rational maps restricted to  $\Gamma_l$  with the marked critical point  $c_{l+1}$  is non-trivial. Theorem 4.2 implies (14). Finally,  $T_{l+1}|_{\Gamma_l}$  is non-zero by Theorem 2.5.  $\square$

Let now  $1 < l \leq 2d-2$ ; since the currents  $T_i$  have continuous local potentials, it is possible to take their wedge product  $T_1 \wedge \cdots \wedge T_l$  (see [De]). This defines a positive closed current of bidegree  $(l, l)$  of locally finite mass, which does not charge analytic subsets. From the previous theorem we get the following corollary, see also [BB2].

**Corollary 8.2.** *For any  $(n_1, \dots, n_l) \in (\mathbb{N}^*)^l$ , choose a collection of integers  $k(n_1) < n_1, \dots, k(n_l) < n_l$ . Let  $W_{n_1, \dots, n_l}$  be the subvariety in  $\text{Rat}_d \setminus \mathcal{L}$  defined as  $W_{n_1, \dots, n_l} = \bigcap_{j=1}^l \text{Per}_j(n_j, k(n_j))$ . Then  $W_{n_1, \dots, n_l}$  has pure codimension  $l$  and*

$$\lim_{n_l \rightarrow \infty} \cdots \lim_{n_1 \rightarrow \infty} \frac{1}{(d^{n_l} + d^{k(n_l)}) \cdots (d^{n_1} + d^{k(n_1)})} [W_{n_1, \dots, n_l}] \rightarrow T_1 \wedge \cdots \wedge T_l.$$

Observe that the same holds when  $\{1, \dots, l\}$  is replaced by any set of  $l$  distinct integers  $\{i_1, \dots, i_l\}$ . The proof is a straightforward adaptation of Corollary 6.17). As a consequence we obtain that near any point of  $\text{Supp}(T_1 \wedge \cdots \wedge T_l)$  there are parameters where  $c_1, \dots, c_l$  are periodic (resp. strictly preperiodic). For  $l = 2d-2$ , these parameters are critically finite (compare with 6.18). Notice that since the currents and measures constructed here do not charge subvarieties, we need not care about the curve  $\mathcal{L}$  of flexible Lattès examples.

Lastly, it follows from the convergence Theorem 4.2 that for each  $j$ ,  $T_j \wedge T_j = 0$ . Indeed,  $T_j$  is the limit of the sequence of hypersurfaces  $\frac{1}{d^n+1} [\text{Per}_j(n, 0)]$ , and  $[\text{Per}_j(n, 0)] \wedge T_j = 0$ . So by using the formulas for the Lyapounov exponent of the maximal entropy measure given in [DeM2, BB1], we can use the same argument as in Section 6 to get that for every  $1 \leq k \leq 2d-2$ , the support of the positive closed  $(k, k)$  current  $(dd^c \text{Lyap}(f))^k$ , studied by [BB1] is accumulated by the set of rational maps where  $k$  critical points are preperiodic.

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